Flatness of C_2-cofinite VOA and Semi-Rigidity

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Setting: $V = \oplus_{n=0}^{\infty} V_n$ is a C_2-cofinite VOA of CFT-type ($\dim V_0 = 1$)
Assume $V \cong V'$ (restricted dual). In my talk, module means f.g.

C_2-Cofiniteness ($\dim V/\langle a_{-2}b \mid a, b \in V\rangle < \infty$) guarantee

(1) Only finitely many simple modules.

(2) All (weak) mod. are $\mathbb{N}$-gradable, ${}^{\exists}N$ s.t. all weights $\in \mathbb{Z}/N\mathbb{Z}$

(3) All intertwining operators have form ($\mathcal{I}_{A,B}^{C}$:the set of int. op.)

$$\mathcal{Y}(w, z) = \sum_{i=0}^{K} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} w_{n,i} z^{-n-1} \log^i z \qquad \text{(logarithmic)}$$

(4) The existence of fusion (tensor) product $\boxtimes$ (as maximal one).

For f.g. V-mod A, B, ${}^{\exists}$f.g V-mod $A \boxtimes B$ and $\mathcal{Y}_{A,B}^{\boxtimes} \in \mathcal{I}_{A,B}^{A\boxtimes B}$ s.t.

${}^{\forall}\mathcal{Y} \in \mathcal{I}_{A,B}^{C}$, ${}^{\exists}\xi \in \mathrm{Hom}(A \boxtimes B, C)$ s.t. $\xi(\mathcal{Y}_{A,B}^{\boxtimes}) = \mathcal{Y}$.

(5) The existence of projective cover of modules.

P is projective (Def: any epi $U \to P$ will split).

(6) Modular invariance. For $S : \tau \to -1/\tau$, the space of (pseudo)-trace functions $\Psi_U^{\phi}(v;\tau) = \mathrm{Tr}_U^{\phi} o(v) e^{2\pi i\tau(L(0)-c/24)}$ is S-invariant.

Main Theorem V C_2-cofinite, CFT-type, $V \cong V'$, Then all f.g. simple V-modules W are flat for fusion products.
i.e. If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact (f.g. V-modules), then
$0 \to A \boxtimes W \xrightarrow{\alpha \boxtimes \mathrm{id}_W} B \boxtimes W \xrightarrow{\beta \boxtimes \mathrm{id}_W} C \boxtimes W \to 0$ is also exact.

Theorem A (Equivalent) If P is a projective cover of V (as a V-module), then a projective cover of W is a direct summand of $P \boxtimes W$.

We can tell "rationality of V" by checking only V

Corollary B V is rational (all modules are completely reducible) iff V is projective as a V-module.

Corollary C

V $\subseteq$ $U \cong U'$ same Virasoro elt.
C_2-cofinite VOA VOA of CFT-type
rational $\Rightarrow$ **U is rational and C_2**

Application to orbifold thoery is: my motivation

Corollary D

$V^{<g>}$ $\subseteq$ V $g \in \mathrm{Aut}(V)$

$\boxed{C_2\text{-cofinite}}$ rational, C_2 finite order $\Rightarrow$ $V^{<g>}$ is rational

I want to take off. (Orbifold conjecture)

The half of proofs of the main theorem are essentially extensions of the proofs of **Verlinde formula** given by Moore-Seiberg and Y.-Z.Huang.

(1) S-transformation diagonalizes the fusion rules. ($\Leftarrow$**Semisimplicity**)

(2) Nonvanishing of the denominators of Verlinde-formula.

i.e. all simple mod. appear in $S(\Psi_V)$, where Ψ_V : trace function on V.

Theorem E This is true under our setting.

A very nice result (Dong-Li-Mason [2000])

Without the assumption of C_2-cofinitness on $V^{<g>}$,

$S(\Psi_{V^{<g>}})$ is a linear sum of parts of twisted-modules.

Outline of the proof:

- ● V C_2-cofiniteness, etc. :

By using Fusing matrix,

$$F(\sigma_{12}(Y^W)\sigma_{13}(Y^W) : Y^W(\sigma_{132}(Y^W)) \neq 0$$

(For rational, see Prop 5.1 in [H](Verlinde formula).)

$\Downarrow$

- ● V-modules are semirigid =Nonvanishing of the denominator of V.F.

Today's talk (a preprint with 14 pages)

projective cover of V

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d, "\boxtimes\ \ id_W"] & & & \\
\phi : & (W\boxtimes W') \arrow[d] & \boxtimes W \arrow[d, "\boxtimes\ \ id_W"] \arrow[rr, "\text{iso}"] & & W \arrow[d, "id_W"'] & \boxtimes(W'\boxtimes W) \arrow[d, "\boxtimes"'] \\
 & V & \boxtimes W(\cong W) & & W & \boxtimes V
\end{tikzcd}$$

$\Downarrow$ (Module theory) (See Preprint server [arxiv0906.1407] 19 pages)

- ● V-modules are flat

We need preliminary results (1)$\sim$ (4). **(Irrational Case)**

(1): Skew int. oporator, Adjoint int. operator (and S_3-action):

For $\mathcal{Y} \in \mathcal{I}_{A,B}^{C}$ (the space of intertwining oprators) ,

$$\begin{aligned} \text{skew op.} \quad \langle c, \sigma_{12}(\mathcal{Y})(b,z)a\rangle &= \langle c, e^{\pi i(h_C-h_A-h_B)}e^{zL(-1)}\mathcal{Y}(a, e^{-\pi i}z)b\rangle, \\ \text{adj. op.} \quad \langle \sigma_{23}(\mathcal{Y})(a,z)c, b\rangle &= \langle c, e^{\pi i h_A}\mathcal{Y}(e^{zL(1)}(e^{-\pi i}z^{-2})^{L(0)}a, z^{-1})b\rangle \end{aligned}$$

Here h_A: conformal weight of A,

wt: semisimple part of $L(0)$ **What is** $(e^{-\pi i}z^{-2})^{L(0)}$ **?**

$$\begin{aligned} (e^{-\pi i}z^{-2})^{L(0)}a : \quad &= e^{(-\pi i - 2\log z)L(0)}a = e^{(-\pi i-2\log z)\mathrm{wt}}e^{(-\pi i - 2\log z)(L(0)-\mathrm{wt})}a \\ &= e^{-\pi \mathrm{wt}(a)}z^{-2\mathrm{wt}(a)}\sum_{m=0}^{\infty}\frac{((-\pi i - 2\log z)(L(0)-\mathrm{wt}))^m}{m!}a \end{aligned}$$

Lemma 1 ([**H**] **Rational**) $\sigma_{12}^2 = \sigma_{23}^2 = 1$ and $\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}$

(**Irrational**) Neither σ_{12} nor σ_{23} are involutions, but $\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}^{-1}\sigma_{12}^{-1}\sigma_{23}$

Denote it by σ_{13}.

Preliminary results (2)   Trace function (by Y.Z.Huang)

Huang has introduced an operator $\mathcal{U}(1)$ satisfying

$$\mathcal{U}(1)\mathcal{Y}(v,x)\mathcal{U}(1)^{-1} = \mathcal{Y}(\mathcal{U}(q^x)v, q^x - 1) = \mathcal{Y}[v,x], \qquad \text{(in Zhu's paper)}$$

where $\mathcal{U}(q^x) := q^{xL(0)}\mathcal{U}(1)$ and $q^x := e^{2\pi i x}$.

c central charge of V. For a V-mod. U and a pseudo-trace ϕ,

$$\mathrm{Tr}_U^{\phi} Y^U(\mathcal{U}(q^y)\mathcal{Y}_{W,W'}^V(w, x-y)w', q^y)q^{\tau(L(0)-c/24)},$$

is absolutely convergent when $0 < |q^\tau| < |q^y| < 1$ and $0 < |q^{x-y} - 1| < 1$.

We extend it to a single valued analytic function

$$\Psi_U^{\phi}(w, w'; x, y, \tau) = E(\mathrm{Tr}_U^{\phi} Y^U(\mathcal{U}(q^y)\mathcal{Y}_{W,W'}^V(w, x-y)w', q^y)q^{\tau(L(0)-c/24)})$$

on a covering $\widetilde{M_1^2}$ of $\{(x,y,\tau) \in \mathbb{C}^2 \times \mathcal{H} \mid x - y \notin \mathbb{Z}\tau + \mathbb{Z}\}$.(We fix pathes.)

We extend the followings for logarithmic intertwining operators.

Lemma 2 For a logarithmic int. op. $\mathcal{Y} \in \mathcal{I}_{A,B}^{C}$, we have
$< c', \sigma_{23}(\mathcal{Y})(\mathcal{U}(x)a, x)b >= e^{-\pi i h_A} < \mathcal{Y}(\mathcal{U}(x^{-1})e^{\pi L(0)}a, x^{-1})c', b >.$

Lemma 3

$$q^{\tau L(0)}\mathcal{Y}(\mathcal{U}(q^y)b, q^y) = \mathcal{Y}(\mathcal{U}(q^{y+\tau})b, q^{y+\tau})q^{\tau L(0)} \qquad (a)$$

$$\mathcal{U}(q^y)\mathcal{Y}(a, x - y)b = \mathcal{Y}(\mathcal{U}(q^x)a, q^x - q^y)\mathcal{U}(q^y)b \qquad (b)$$

Preliminary results (3) Three transformations:

(i) S-transformation $(\tau \to -1/\tau)$

$S(\Psi_U^{\phi})\,(w,\tilde{w};x,y,\tau) := \Psi_U^{\phi}\left(\left(\frac{-1}{\tau}\right)^{\mathrm{wt}} w,\, \left(\frac{-1}{\tau}\right)^{\mathrm{wt}} \tilde{w};\, \frac{-1}{\tau}x,\, \frac{-1}{\tau}y;\, \frac{-1}{\tau}\right)$.

(ii) Huang's two transformations with slight changes.

$\alpha(\Psi_W^{\phi}(w,w':x,y,\tau)) := \Psi_W^{\phi}(w,w':x,y+1,\tau) \quad y+1=\lim_{0<t\to 1}(y+t)$

$\beta(\Psi_W^{\phi}(w,w':x,y,\tau)) := \Psi_W^{\phi}(w,w':x,y,\tau) \quad y+\tau=\lim_{0<t\to 1}(y+t\tau)$

Remark: We don't use projections to the actons of V.

Lemma 4 $S\alpha=\beta S$

In the proof of Verlinde formula:

$\alpha \Rightarrow$diagonal matrix, $\beta \Rightarrow$Fusion rules.

One result: Since $\dim \mathrm{Hom}(W\boxtimes W'\boxtimes V, V)=1$, $\exists\lambda\neq 0$ s.t.

$\alpha(\Psi_V(w,w':x,y,\tau))=\lambda\Psi_V(w,w':x,y,\tau)$.

$S(\Psi_V(w,w':x,y,\tau)$ is β-invariant (upto scalar times).

A collaboration of β and trace-function:

$$\begin{array}{ll}
 & \mathrm{Tr}_U^{\phi}\mathcal{Y}_{W',E}^{U}\mathcal{Y}_{W,U}^{E}q^{\tau(L(0)-c/24)} \\
\text{action of } \beta\Rightarrow & \mathrm{Tr}_U^{\phi}\mathcal{Y}_{W',E}^{U}\mathcal{Y}_{W,U}^{E}(\mathcal{U}(q^{\tau}))q^{\tau(L(0)-c/24)} \\
\text{Lemma 3 }\Rightarrow & =\mathrm{Tr}_U^{\phi}\mathcal{Y}_{W',E}^{U}q^{\tau L(0)}\mathcal{Y}_{W,U}^{E}q^{\tau(-c/24)} \\
\text{(pseudo)trace is symmetric}\Rightarrow & =\mathrm{Tr}_E^{\phi'}\mathcal{Y}_{W,U}^{E}\mathcal{Y}_{W',E}^{U}q^{\tau L(0)}2^{2\pi i\tau(-c/24)}
\end{array}$$

So it replaces U by a middle of compositions of two int.op.

Outline of proof of the main theorem.

Suppose fale and let W be not semi-rigid. We will show

$$\beta(E(\mathrm{Tr}^{\phi}_{U} Y^{U}(\mathcal{U}(q^{x})\mathcal{Y}^{V}_{W,W'}(a, x-y)b, q^{y})q^{\tau(L(0)-c/24)}))$$

has no parts given by $V \cong W \boxtimes W'/\ker\langle\cdot,\cdot\rangle$ for any U. (1)

On the other hand, since

(2) $S(\Psi_{V}(w, w' : x, y, \tau))$ is a $\mathbb{C}[\tau]$-linear combination of $E(\mathrm{Tr}^{\phi}_{U} Y^{U}(\mathcal{U}(q^{x})\mathcal{Y}^{V}_{W,W'}(a, x-y)b, q^{y})q^{\tau(L(0)-c/24)})$, which are given by V.

(3) $S(\Psi_{V}(w, w' : x, y, \tau))$ is β-invariant (upto scalar times).

This conflicts with (1).

Preliminary results (4) For the proof, we will use

Fusing Matrix(by Moore-Seiberg and Y.Z.Huang)

For $a \in A, b \in B, c \in C, d \in D$, a composition of intertwining operators

$$\langle d, \mathcal{Y}_1(a,x)\mathcal{Y}_2(b,y)c\rangle$$

is absolutely convergent in $|x| > |y| > 0$. (Note: $0 < \arg(z^{1/N}) < 2\pi/N$)

Extend it analytically to a single valued function

$$E(\langle d, \mathcal{Y}_1(a,x)\mathcal{Y}_2(b,y)c\rangle)$$

on a covering $\widetilde{M}$ of $\{(x,y) \in \mathbb{C}^2 \mid xy(x-y) \neq 0\}$. (i.e. We fix pathes)

Similarly, we can define

$$E(\langle d, \mathcal{Y}_4(\mathcal{Y}_3(a,x-y)b,y)c\rangle)$$

They span the same space. For bases $\{\mathcal{Y}_{1j}\mathcal{Y}_{2j} \mid i\}$,$\{\mathcal{Y}_{4i}(\mathcal{Y}_{3i}) \mid i\}$,

$$\begin{aligned}&E(\langle d, \mathcal{Y}_{1j}(a,x)\mathcal{Y}_{2j}(b,y)c\rangle\\ &\qquad = \sum F(\mathcal{Y}_{1j}\mathcal{Y}_{2j} : \mathcal{Y}_{4i}(\mathcal{Y}_{3i}))E(\langle d, \mathcal{Y}_{4,i}(\mathcal{Y}_{3,i}(a,x-y)b,y)c\rangle)\end{aligned}$$

Fusing matrix

Relation among fusing matrices 1

Fusing matrix corresponds 4-cycle $(1,2,3,4)$ on A,B,C,D. (S_4-action.)
From the direct calculation, we have:

$$F(\mathcal{Y}_1\mathcal{Y}_2 : \mathcal{Y}^{i1}(\mathcal{Y}^{i2})) = F^{-1}(\sigma_{12}(\mathcal{Y}_1)(\sigma_{12}(\mathcal{Y}_2)) : \sigma_{12}(\mathcal{Y}^{i1})\sigma_{12}(\mathcal{Y}^{i2}))_{\Omega} \quad (1)$$

where $\Omega = \{\sigma_{12}(\mathcal{Y}^{i1})\sigma_{12}(\mathcal{Y}^{i2}) : i\}$ is a new basis.
This corresponds to $(12)(34)(1234)(12)(34) = (2143) = (1234)^{-1}$.

Similarly, we have: for a new basis $\Omega' = \{\sigma_{123}(\mathcal{Y}^{i1})(\sigma_{132}(\mathcal{Y}^{i2})) \mid i\}$

$$F(\mathcal{Y}_1\mathcal{Y}_2, \mathcal{Y}^{i1}(\mathcal{Y}^{i2})) = F(\sigma_{132}(\mathcal{Y}_2)\sigma_{123}(\mathcal{Y}_1) : \sigma_{123}(\mathcal{Y}^{i2})(\sigma_{132}(\mathcal{Y}^{i1})))_{\Omega'} \quad (2)$$

where $\sigma_{123} = \sigma_{12}\sigma_{23}, \sigma_{132} = \sigma_{23}\sigma_{12}$.

New notatoin of Bases: $a \in A, b \in B, c \in C, d \in D,\ \mathcal{Y}_{3,i} \in \mathcal{I}^{E}_{A,B},$

Similarly, the left side is $E(\langle d, \sigma_{132}(\mathcal{Y}^{\boxtimes}_{D,A})(a,x)\mu\mathcal{Y}^{\boxtimes}_{B,C}(b,y)c\rangle)$

$E(\langle d, \mathcal{Y}_1(a,x)\mathcal{Y}_2(b,y)c\rangle$ $= \sum_i \langle d, \mathcal{Y}_{4,i}(\mathcal{Y}_{3,i}(a,x-y)b,y)c\rangle)$

because of fusion product $\uparrow \xi_i : A \boxtimes B \to E$

$^{\exists}(\mathcal{Y}^{\boxtimes}_{A,B}(a,x-y)b$

$= \sum_i E(\langle d, \mathcal{Y}_{4,i}\xi_i(\mathcal{Y}^{\boxtimes}_{A,B}(a,x-y)b,y)c\rangle)$

similarly $^{\exists}\xi'_i$ s.t. $= \sum_i E(\langle d, \sigma_{123}(\mathcal{Y}^{\boxtimes}_{C,D})\xi'_i\xi_i\mathcal{Y}^{\boxtimes}_{A,B}(a,x-y)b,y)c\rangle)$

$= E(\langle d, \sigma_{123}(\mathcal{Y}^{\boxtimes}_{C,D})(\sum_i \xi'_i\xi_i)\mathcal{Y}^{\boxtimes}_{A,B}(a,x-y)b,y)c\rangle)$

where $\sum \xi'_i\xi_i \in \mathrm{Hom}(A \boxtimes B, (C \boxtimes D)'),\ \mu \in \mathrm{Hom}(B \boxtimes C, (D \boxtimes A)')$

$\sigma_{123} = \sigma_{12}\sigma_{23}, \sigma_{132} = \sigma_{23}\sigma_{12}$

Use a basis $\{\xi_j \mid j\}$ of $\mathrm{Hom}(A \boxtimes B, (C \boxtimes D)')$ to denote a basis of

$$E(\langle d, \sigma_{123}(\mathcal{Y}^{\boxtimes}_{C,D})\xi_j\mathcal{Y}^{\boxtimes}_{A,B}(a,\cdot)b,\cdot)c\rangle)$$

Relation among fusing matrices 2

One advantage of our basis is:

For $\mathcal{Y}^{\boxtimes}_{W,U'}$, $\mathcal{Y}^{\boxtimes}_{W\times U',T}$ and $\mathcal{Y}^{\boxtimes}_{U',T}$, we choose $\mathcal{Y}^{\boxtimes}_{W,U'\boxtimes T}$ so that

$$\langle d', \mathcal{Y}^{\boxtimes}_{W\boxtimes U',T}(\mathcal{Y}^{\boxtimes}_{W,U'}(a,x-y)b,y)c\rangle = \langle d', \mathcal{Y}^{\boxtimes}_{W,U'\boxtimes T}(a,x)\mathcal{Y}^{\boxtimes}_{U',T}(b,y)c\rangle.$$

Then the following equation holds for any m:

$$\begin{aligned}&\textstyle\sum_k F(\mathcal{Y}^{W}_{W,V}\mathcal{Y}^{V}_{U',U}:\sigma_{13}(\mathcal{Y}^{\boxtimes}_{W',U})\xi_k\mathcal{Y}^{\boxtimes}_{W,U'})F(\mathcal{Y}^{W}_{W\boxtimes U',U:k}\mathcal{Y}^{U}_{T,W}:\mathcal{Y}_{W\boxtimes U'\times T,W:0}(\mathcal{Y}^{\boxtimes}_{W\boxtimes U',T}))\\ =&\textstyle\sum_s F(\mathcal{Y}^{V}_{U',U}\mathcal{Y}^{U}_{T,W}:\mathcal{Y}^{V}_{U'\times T,W:s}(\mathcal{Y}^{\boxtimes}_{U',T}))\underbrace{F(\mathcal{Y}^{W}_{W,V}\mathcal{Y}^{W}_{U'\boxtimes T,W:s}:\mathcal{Y}^{W}_{W\times U'\boxtimes T,W:0}(\mathcal{Y}^{\boxtimes}_{W,U'\times T}))}_{\text{If nonsemirigid, then } =0}.\end{aligned}$$

If we choose $m=0$ s.t. $\mathcal{Y}^{W}_{W\times U'\boxtimes T,W:m=0}(\mathcal{Y}^{\boxtimes}_{W,U'\times T}) = \mathcal{Y}^{W}_{V,W}(\mathcal{Y}^{V}_{W,U'\times T})$

Then

Lemma 5. If W is not semirigid, then

$$\textstyle\sum_k F(\mathcal{Y}^{W}_{W,V}\mathcal{Y}^{V}_{U',U}:\mathcal{Y}^{W}_{W\boxtimes U',U;k}\mathcal{Y}^{\boxtimes}_{W,U'})F(\mathcal{Y}^{W}_{W\boxtimes U',U;k}\mathcal{Y}^{U}_{T,W}:\mathcal{Y}_{W\boxtimes U'\times T,W:0}(\mathcal{Y}^{\boxtimes}_{W\boxtimes U',T})) = 0$$

Set $q^* = q^{\tau(L(0)-c/24)}$. Actually, by the direct calculation, we have

$$\begin{aligned}
&\beta(E(\mathrm{Tr}_U^{\phi} Y^U(\mathcal{U}(q^y)\mathcal{Y}_{W,W'}^{V}(a, x-y)b, q^y)q^*))\\
&= E(\mathrm{Tr}_U^{\phi} Y^U(\mathcal{U}(q^{y+\tau})\mathcal{Y}_{W,W'}^{V}(a, x-(y+\tau))b, q^{y+\tau})q^*) \qquad \text{Def of } \beta\\
&= E(\mathrm{Tr}_U^{\phi} Y^U(\mathcal{Y}_{W,W'}^{V}(\mathcal{U}(q^x)w, q^x - q^{y+\tau})\mathcal{U}(q^{y+\tau})w', q^{y+\tau})q^*) \ \text{Lem.3 (b)}\\
&= \textstyle\sum_k F^{-1}(Y^U(\mathcal{Y}_{W,W'}^{V}) : \sigma_{23}(\mathcal{Y}_{V,U'}^{\boxtimes})\xi_k \cdot \mathcal{Y}_{W',U}^{\boxtimes}) \times \qquad \text{Def of } F^{-1},\ q^{y+\tau} = q^y q^\tau\\
&\qquad \times E(\mathrm{Tr}_U^{\phi} \sigma_{23}(\mathcal{Y}_{W,U'}^{\boxtimes})(\mathcal{U}(q^x)a, q^x)\xi_k \mathcal{Y}_{W',U}^{\boxtimes}(\mathcal{U}(q^\tau q^y)b, q^{y+\tau})q^*)\\
&= \textstyle\sum_k F^{-1}(Y^U(\mathcal{Y}_{W,W'}^{V}) : \sigma_{23}(\mathcal{Y}_{W,U'}^{\boxtimes})\xi_k \cdot \mathcal{Y}_{W',U}^{\boxtimes}) \times \qquad \text{Lem.3 (a)}\\
&\qquad \times E(\mathrm{Tr}_U^{\phi} \sigma_{23}(\mathcal{Y}_{W,U'}^{\boxtimes})(\mathcal{U}(q^x)a, q^x)q^* \xi_k \mathcal{Y}_{W',U}^{\boxtimes}(\mathcal{U}(q^y)b, q^y))\\
&= \textstyle\sum_k F^{-1}(Y^U(\mathcal{Y}_{W,W'}^{V}) : \sigma_{23}(\mathcal{Y}_{W,U'})\xi_k \mathcal{Y}_{W',U}^{\boxtimes}) \times \qquad \mathrm{Tr}^{\phi} \text{ is symmetric}\\
&\ \times E(\mathrm{Tr}_{(W\boxtimes U')'}^{\phi} \xi_k \mathcal{Y}_{W',U}^{\boxtimes}(\mathcal{U}(q^y)b, q^y)\sigma_{23}(\mathcal{Y}_{W,U'}^{\boxtimes})(\mathcal{U}(q^x)a, q^x)q^*)\\
&= \textstyle\sum_k F^{-1}(Y^U(\mathcal{Y}_{W,W'}^{V}) : \sigma_{23}(\mathcal{Y}_{W,U'}^{\boxtimes})\xi_k \mathcal{Y}_{W',U}^{\boxtimes})e^{-\pi i(h_2 + h_{2'})} \times \qquad \text{Lem.4}\\
&\ \times E(\mathrm{Tr}_{W\boxtimes U'}^{\phi'} \mathcal{Y}_{W,U'}^{\boxtimes}(\mathcal{U}(q^{-x})e^{\pi i \mathrm{wt}}a, q^{-x})\sigma_{23}(\xi_k \mathcal{Y}_{W',U}^{\boxtimes})(\mathcal{U}(q^{-y})e^{\pi i \mathrm{wt}}b, q^{-y})q^*)\\
&= \textstyle\sum_m \sum_k F^{-1}(Y^U(\mathcal{Y}_{W,W'}^{V}) : \sigma_{23}(\mathcal{Y}_{W,U'}^{\boxtimes})\xi_k \mathcal{Y}_{W',U}^{\boxtimes}) \times \qquad \text{Def of } F\\
&\ \times F(\mathcal{Y}_{W,U'}^{\boxtimes})\sigma_{23}(\xi_k \mathcal{Y}_{W',U}^{\boxtimes}) : \sigma_{123}(\mu_m \mathcal{Y}_{W\times W',(W\times U')'}^{\boxtimes}(\mathcal{Y}_{W,W'}^{\boxtimes}))e^{-2\pi i h_2} \times\\
&\ \times E(\mathrm{Tr}_{(W\boxtimes U')}^{\phi'} \sigma_{123}(\mu_m \mathcal{Y}_{W\times W',(W\times U')'}^{\boxtimes}(\mathcal{U}(q^{-y})\mathcal{Y}_{W,W'}^{\boxtimes}(e^{\pi i \mathrm{wt}}a, e^{\pi i}(x-y))b, q^{-y})q^*).
\end{aligned}$$

We may assume that $\sigma_{123}(\mu_0\mathcal{Y}^{\boxtimes}_{W\times W',(W\times U')'})(\mathcal{Y}^{\boxtimes}_{W,W'})$ is given by the action of V, that is, $= \mathcal{Y}^{W\times U'}_{W\times W',W\times U'}(\mathcal{Y}^{V}_{W,W'})$,

then the coefficient at μ_0 is

$$\sum_k F^{-1}(Y^U(\mathcal{Y}^V_{W,W'}) : \sigma_{23}(\mathcal{Y}^\boxtimes_{W,U'}) \cdot \xi_k\mathcal{Y}^\boxtimes_{W',U})\times$$

$$\times F(\mathcal{Y}^\boxtimes_{W,U'}\sigma_{23}(\xi_k\mathcal{Y}^\boxtimes_{W',U}) : \sigma_{123}(\mu_0\mathcal{Y}^\boxtimes_{W\times U',(W\times U')'})(\mathcal{Y}^\boxtimes_{W,W'}))$$

$$= \sum_k F(\sigma_{12}(Y^U)\sigma_{12}(\mathcal{Y}^V_{W,W'}) : \sigma_{12}(\sigma_{23}(\mathcal{Y}^\boxtimes_{W,U'})(\sigma_{12}(\xi_k\mathcal{Y}^\boxtimes_{W',U}))) \times \quad \text{by } \sigma_{12}, \sigma_{123}^{-1}$$

$$\times F(\sigma^{-1}_{123}\sigma_{23}(\xi_k\mathcal{Y}^\boxtimes_{W',U})\sigma^{-1}_{132}(\mathcal{Y}^\boxtimes_{W,U'}) : \sigma^{-1}_{132}(\mathcal{Y}^\boxtimes_{W,W'})(\sigma^{-1}_{123}\sigma_{123}(\mu_0\mathcal{Y}^\boxtimes_{W\times U',(W\times U')'})))$$

$$= \sum_k F(\sigma_{132}\sigma_{12}(\mathcal{Y}^V_{W,W'})\sigma_{123}\sigma_{12}(Y^U) : \sigma_{123}\sigma_{12}(\xi_k\mathcal{Y}^\boxtimes_{W',U})(\sigma_{132}\sigma_{123}(\mathcal{Y}^\boxtimes_{W,U'})))\times$$

$$\times F(\sigma_{13}(\xi_k\mathcal{Y}^\boxtimes_{W',U})\sigma^{-1}_{132}(\mathcal{Y}^\boxtimes_{W,U'}) : \sigma^{-1}_{132}(\mathcal{Y}^\boxtimes_{W,W'})(\mu_0\mathcal{Y}^\boxtimes_{W\times U',(W\times U')'})) \quad \text{by } \sigma_{132}$$

$$= \sum_k \lambda F(\mathcal{Y}^W_{W,V}\mathcal{Y}^V_{U',U} : \sigma_{13}(\xi_k\mathcal{Y}^\boxtimes_{W',U})(\mathcal{Y}^\boxtimes_{W,U'})) \times \quad \text{Rewrite}$$

$$\times F(\sigma_{13}(\xi_k\mathcal{Y}^\boxtimes_{W',U})\sigma^{-1}_{132}(\mathcal{Y}^\boxtimes_{W,U'}) : \sigma^{-1}_{132}(\mathcal{Y}^\boxtimes_{W,W'})(\mu_0\mathcal{Y}^\boxtimes_{W\times U',(W\times U')'}))$$

$$= 0 \qquad \text{by Lem.5}$$

This completes the proof of the main theorem.

Note: Lem.5.

$$\sum_k F(\mathcal{Y}^W_{W,V}\mathcal{Y}^V_{U',U} : \mathcal{Y}^W_{W\boxtimes U',U;k}\mathcal{Y}^\boxtimes_{W,U'})F(\mathcal{Y}^W_{W\boxtimes U',U;k}\mathcal{Y}^U_{T,W} : \mathcal{Y}_{W\boxtimes U'\times T,W:0}(\mathcal{Y}^\boxtimes_{W\boxtimes U',T}))$$

$$= 0$$

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