Vector-Valued Modular Forms and Logarithmic CFT

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October 16, 2009

1 Introduction

a) The large majority of studies of vertex operator algebras V assume two properties of V: *CFT-type* and *rational*. That is,

$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus \ldots$
The category of (admissible) V-modules is *semisimple*.

This situation derives from physics: unitarity, and the attempt to understand the idea of rational conformal field theory.

b) Recent developments suggest that more general settings must be considered.

(i) Work of Gorbounov-Malikov-Schechtman-Vaintrob (cf. [MSV]) on the *chiral de Rham complex*. Here, it is important to consider the case when the vacuum may be degenerate, so that

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \ldots$$

where we assume only $\dim V_0 \geq 1$. It is well-known that with this set-up, the product

$$a \otimes b \mapsto a_{-1}b$$

equips V_0 with the structure of a commutative, associative, unital algebra. One can informally think of V_0 as the (even) cohomology ring of some compact manifold associated to V.

(ii) The advent of *logarithmic field theory.* This corresponds VOAs with only finitely many ordinary irreducible modules, but completely reducibility of modules is *not assumed.* One way to ensure such a situation is to assume C_2*-cofiniteness*:

$$\dim(V/\langle u_{-2}v \mid u, v \in V\rangle) < \infty.$$

c) There are two questions that naturally arise here. Are the implications

$$\text{rational} \Rightarrow C_2\text{-cofinite},$$
$$\text{logarithmic} \Rightarrow C_2\text{-cofinite},$$

always true? In practice, one usually assumes either C_2-cofiniteness or *both* rationality and C_2-cofiniteness. We shall do this.

d) V may, or may not, carry a *nonzero invariant bilinear form*

$$(\ ,\ ) : V \otimes V \to \mathbb{C}.$$

Recall that invariant means that

$$(Y(a, z)b, c) = (b, Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})c).$$

If V has such a form, and if V is CFT-type, then $\ker(\ ,\ )$ is the unique maximal ideal of V. In this case, then, V is simple if, and only if, $(\ ,\ )$ is *nondegenerate.*

e) In this talk we will consider simple, nondegenerate VOAs V which are C_2-cofinite and satisfy

$$V = V_0 \oplus V_1 \oplus \dots, \quad \dim V_0 \geq 1.$$

In particular, V_0 is equipped with a nondegenerate pairing, and we may think of V as being associated to an *oriented manifold.*

If V is also rational with

$$\dim V_0 = 1,$$

we say that V is *strongly regular.*

f) There is the beginning of a *structure theory for strongly regular VOAs.* This includes the following results ([DM1], [DM2]):

I: *Reductivity*: the Lie algebra on V_1 is reductive:

$$V_1 = \text{abelian} \perp L_1 \perp \ldots \perp L_t$$

with simple Lie algebras L_i.

II: *Integrability*: The fields $Y(v, z)$, $v \in L_i$, close on the *affine Lie algebra* $\widehat{L_i}$ corresponding to the simple VOA $L(k_i, 0)$ of *positive integral level* k_i; V is an integrable $\widehat{L_i}$-module.

We want to extend this theory to the case $\dim V_0 \geq 1$, assuming C_2-cofiniteness but not necessarily rationality. We concentrate on the extension of I in this talk.

2 Examples

a) There are only a few examples of C_2-cofinite VOAs with $\dim V_0 > 1$ in the literature. They are constructed by the method of *shifting* ([DM3]).

Shifting means this: start with a VOA $V = (V, Y, \mathbf{1}, \omega)$, then *shift* the Virasoro element to another one ω', without changing the state space V or any of the fields $Y(v, z)$. Then we can sometimes get a shifted VOA $(V, Y, \mathbf{1}, \omega')$ which often has surprising properties.

Generally, one expects to find many Virasoro states ω' in V, so there is lots of scope for creativity. Here's an example.

b) Suppose that $h \in V_1$ satisfies

$$L_1 h + h_1 h = \lambda \mathbf{1}.$$

Set $\omega_h = \omega + h_{-2}\mathbf{1}$. Then ω_h is a Virasoro element of central charge $c_h = c - 12\lambda$:

$$\begin{aligned} Y(\omega_h, z) &= \sum_{n \in \mathbb{Z}} L_h(n) z^{-n-2} \\ [L_h(m), L_h(n)] &= (m-n)L_h(m+n) + \delta_{m,-n}\frac{m^3 - m}{12} c_h Id. \end{aligned}$$

Additional integrality conditions are required to ensure that $(V, Y, \mathbf{1}, \omega_h)$ is a VOA.

c) Take L to be an even lattice and $V = V_L$ the corresponding lattice theory. Let

$$\Phi = \{\alpha \in L \mid (\alpha, \alpha) = 2\}.$$

Φ is a semisimple *root system* whose associated semisimple Lie algebra L_Φ has simple components of type ADE, and

$$(V_L)_1 = \text{abelian} \oplus L_\Phi.$$

$L \subseteq L^0 \subseteq H \subseteq (V_L)_1$ with H a *Cartan subalgebra.* Here, L^0 is the *dual* lattice. Choose

$$h \in L^0.$$

Then the following hold ([DM3]):

$$V_{L,h} = (V_L, Y, \mathbf{1}, \omega_h)$$

is a VOA. Moreover

$$\begin{aligned}
&V_{L,h} \text{ is nondegenerate} \Leftrightarrow 2h \in L, \\
&V_{L,h} \text{ has no negative weight spaces} \Leftrightarrow (\alpha + h, \alpha + h) \geq (h, h)(\alpha \in L), \\
&(V_{L,h})_0 = \langle e^{-\alpha} \mid (\alpha + h, \alpha + h) = (h, h)\rangle \supseteq \{\mathbf{1}, e^{-2h}, \ldots\}, \\
&\dim(V_{L,h})_0 \geq 2 \Leftrightarrow h \neq 0.
\end{aligned}$$

3 1-truncation

a) The axioms for V give rise to an algebraic structure on the 1-truncation of V, i.e. the pair (V_0, V_1). In particular,

$$\begin{aligned}
&V_0 \text{ is a commutative algebra} \\
&(a_0 b)_0 c + (c_0 a)_0 b + (b_0 c)_0 a = 0 \quad (a, b, c \in V_1), \\
&a_0 b + b_0 a = L_{-1} a_1 b \quad \text{for } a, b, c \in V_1.
\end{aligned}$$

Hence, if $N = L_{-1} V_0 \subseteq V_1$ then

$$V_1/N \text{ is naturally a Lie algebra.}$$

If V is of CFT-type then $N = 0$ and V_1 is a Lie algebra. But in the more general case V_1 is generally *not* a Lie algebra.

In the case of shifted lattice theories, for example, $(V_{L,h})_1$ is not a Lie algebra if $h \neq 0$. Moreover in this case

$$(V_{L,h})_1/N \text{ is } \textit{not} \text{ reductive.}$$

So in general, the Lie structures related to V_1 will be more complicated when $\dim V_0 > 1$, compared to the case of CFT-type.

We want to find an intrinsic description of the nil radical of V_1/N.

4 The special state t

a) Take any VOA

$$V = V_0 \oplus V_1 \oplus \ldots.$$

The natural commutative associative algebra on V_0 has a canonical decomposition as a direct sum of *local algebras*

$$V_0 = e_1 V_0 \oplus \ldots \oplus e_r V_0.$$

(A local algebra has Jacobson radical $J(V_0)$ of codimension 1.) The e_i are the *primitive idempotents* of V_0.

There is a corresponding decomposition of V into *local VOAs* ([DM4]):

$$V = e_1(-1)V \oplus \ldots \oplus e_r(-1)V.$$

(By definition, a local VOA has a unique maximal proper ideal.) The main point is that the idempotents e_i are *central elements* of V, i.e.

$$Y(e_i, z) = e_i(-1).$$

b) Now assume that V is *simple*. The $e_i(-1)V$ are ideals in V, so $r = 1$. Hence

$$V \text{ simple} \Rightarrow V_0 \text{ is a } \textit{local algebra}.$$

c) Now assume in addition that V has a nondegenerate form $(\ ,\ )$. Then V_0 has a nondegenerate invariant bilinear form, ie. it is a *symmetric or Frobenius* algebra. Hence, the map

$$I \mapsto I^{\perp}$$

is an inclusion reversing bijection between ideals $I \subseteq V_0$. In particular, V_0 has a *unique minimal* ideal $T = J(V_0)^{\perp}$ and $\dim T = 1$. Nondegeneracy implies $(T, \mathbb{C}\mathbf{1}) \neq 0$, so there is a *unique* state $t \in T$ with

$$(t, \mathbf{1}) = 1.$$

In the cohomological heuristic, t spans the top nonvanishing cohomology group and is Poincaré dual to $\mathbf{1}$.

d) Introduce $\langle\ ,\ \rangle : V_1 \otimes V_1 \to \mathbf{C}$:

$$\langle u, v \rangle = (u_1 v, t).$$

If $a \in V_0$, one shows that

$$(L_{-1}a)_1 v \in J(V_0), v \in V_1.$$

Therefore

$$\langle L(-1)a, v \rangle = ((L(-1)V_0)(1)v, t) = 0,\ \ a \in V_0,\ \ v \in V_1$$

$$\Rightarrow \qquad\qquad N = L(-1)V_0 \subseteq \ \mathrm{Rad}\langle\ ,\ \rangle.$$

It turns out that $\langle\ ,\ \rangle$ induces a well-defined *symmetric, invariant, bilinear form*

$$\widetilde{\langle\ ,\ \rangle} : V_1/N \otimes V_1/N \to \mathbb{C}.$$

Conjecture: Suppose that V is C_2-cofinite, simple and nondegenerate. Then

$$\mathrm{Rad}\widetilde{\langle\ ,\ \rangle} = \mathrm{Nil}(V_1/N).$$

e) Evidence.

(i) True for all $V_{L,h}$ with $2h \in L$ (by a calculation).

(ii) True if V is strongly regular. In this case $V_0 = \mathbb{C}\mathbf{1} \Rightarrow t$ is a scalar multiple of $\mathbf{1}$, $\langle\ ,\ \rangle$ a scalar multiple of $(\ ,\ )$ and therefore nondegenerate, $N = 0$, and the Conjecture says V_1 is reductive. This is proved in [DM1].

Theorem: If V is C_2-cofinite, simple and nondegenerate then

$$\mathrm{Nil}(V_1/N) \subseteq \mathrm{Rad}\widetilde{\langle\ ,\ \rangle} \subseteq Solv(V_1/N).$$

This is part of ongoing joint work with Gail Yamskulna ([MY]).

We will indicate how to prove

$$\mathrm{Nil}(V_1/N) \subseteq \mathrm{Rad}\widetilde{\langle\ ,\ \rangle}.$$

This already implies that the Conjecture holds if $\dim V_0 = 1$, even without the assumption of rationality.

5 The rational case

a) Here is how to prove that $\mathrm{Nil}(V_1/N) \subseteq \mathrm{Rad}\widetilde{\langle\ ,\ \rangle}$ if V is also assumed to be *rational*. It depends on *modular-invariance*.

b) Let the ordinary irreducible modules of V be $M^1, \ldots, M^r$, with

$$Z_{M^j}(v, \tau) = \mathrm{Tr}_{M^j}\, o(v) q^{L(0)-c/24}.$$

Take $u, v \in V_1$ with $v + N \in \mathrm{Nil}(V_1/N)$. Must show $\langle u, v\rangle = 0$. If not, may assume $\langle u, v\rangle = (u_1 v, t) = 1$. So

$$u_1 v = \mathbf{1} + x,\ x \in J(V_0).$$

c) Consider Zhu's identity

$$\mathrm{Tr}_{M^j}\, o(u) o(v) q^{L(0)-c/24} =$$
$$Z_{M^j}(u_{[-1]}v, \tau) - \sum_{k\geq 1} E_{2k}(\tau) Z_{M^j}(u_{[2k-1]}v, \tau).$$

$o(u)o(v)$ is a nilpotent operator on each M^j, so the left-hand-side of this equation vanishes. $o(x)$ is also nilpotent. Also $u_{[2k-1]}v = 0$ for $k > 1$ and $u_{[1]}v = u_1 v$. Thus

$$Z_{M^j}(u_{[-1]}v, \tau) = E_2(\tau) Z_{M^j}(\mathbf{1}, \tau).$$

Modular-invariance in the rational case says this: if $v \in V_{[k]}$ and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

then

$$\begin{aligned} Z_{M^i}(v,\tau)|_k\gamma &:= (c\tau + d)^{-k} Z_{M^i}(v, \gamma\tau) \\ &= \sum_{j=1}^{r} \rho(\gamma)_{ij} Z_{M^j}(v,\tau) \end{aligned}$$

for a representation $\rho : SL_2(\mathbb{Z}) \to GL_r(\mathbb{C})$ independent of v, τ. Now take u, v as before, $\gamma = S$. Then

$$\begin{aligned} 0 &\neq \left(-\frac{1}{2\pi i\tau}\right) Z_{M^i}(\mathbf{1}, S\tau) \\ &= \left(-\frac{1}{2\pi i\tau}\right) \sum_{j=1}^{r} \rho(S)_{ij} Z_{M^j}(\mathbf{1},\tau) + \sum_{j=1}^{r} \rho(S)_{ij} Z_{M^j}(u_{[-1]}v, \tau) \\ &\qquad - \sum_{j=1}^{r} \rho(S)_{ij} Z_{M^j}(u_{[-1]}v, \tau) \\ &= \left(-\frac{1}{2\pi i\tau} + E_2(\tau)\right) \sum_{j=1}^{r} \rho(S)_{ij} Z_{M^j}(\mathbf{1},\tau) \\ &\quad - \sum_{j=1}^{r} \rho(S)_{ij} Z_{M^j}(u_{[-1]}v, \tau) \\ &= \tau^{-2} E_2(S\tau) Z_{M^i}(\mathbf{1}, S\tau) - \sum_{j=1}^{r} \rho(S)_{ij} Z_{M^j}(u_{[-1]}v, \tau) \\ &= \tau^{-2} Z_{M^i}(u_{[-1]}v, S\tau) - \sum_{j=1}^{r} \rho(S)_{ij} Z_{M^j}(u_{[-1]}v, \tau) \\ &= Z_{M^i}(u_{[-1]}v, \tau)|_2 S - \sum_{j=1}^{r} \rho(S)_{ij} Z_{M^j}(u_{[-1]}v, \tau) \\ &= 0, \end{aligned}$$

which gives the desired contradiction.

6 The C_2-cofinite case

a) Let V be a C_2-cofinite simple VOA. There are a finite number of ordinary irreducible V-modules, say $M^1, \ldots, M^r$.

The extension of Zhu's modular-invariance to C_2-cofinite VOAs has been established by Masahiko Miyamoto [M]. It is more difficult to explain, in part because L_0 is no longer a semisimple operator on V-modules. Let

$$\begin{aligned}
\mathbb{H} &= \text{complex upper half-plane}, \\
\mathfrak{F} &= \{\text{holomorphic functions } f : \mathbb{H} \to \mathbb{C}\}, \\
\Gamma &= SL_2(\mathbb{Z}), \\
\gamma\tau &= \frac{a\tau + b}{c\tau + d} \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \tau \in \mathbb{H}\right), \\
f|_k\gamma(\tau) &= (c\tau + d)^{-k} f(\gamma\tau) \quad (f \in \mathfrak{F}), \\
\mathfrak{F}_k &= \text{right } \Gamma\text{-module } \mathfrak{F} \text{ with respect to } |_k, \\
q &= e^{2\pi i\tau}.
\end{aligned}$$

A weak version of modular-invariance for C_2-cofinite VOAs in this situation can now be stated as follows.

> Let $v \in V_{[k]}$. The trace functions $Z_{M^j}(v, \tau)$ generate a finite-dimensional Γ-submodule of $\mathfrak{F}_k$.

In general the trace functions do *not span* a Γ-submodule. Rather, one has q-expansions

$$f_{ij}(\tau) = \sum_{n \geq 0} a_{ij}(n) q^{n+\mu_i}$$

such that

$$Z_{M_j}(v)|_k\gamma(\tau) = \sum_{i=1}^{n} \tau^i f_{ij}(\tau).$$

If you are not used to seeing them, the powers of τ may look strange!

Using a more precise version of modular-invariance, we can then argue much as in the rational case to prove the desired containment, namely

$$\mathrm{Nil}(V_1/N) \subseteq \mathrm{Rad}\widetilde{\langle\ ,\ \rangle}.$$

7 Logarithmic vector-valued modular forms

a) We start by explaining why one should expect powers of τ along with q-expansions when dealing with trace functions in logarithmic field theories.

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $W \subseteq \mathfrak{F}_k$ be a finite-dimensional $\langle T \rangle$-module. T acts by

$$f(\tau) \mapsto f|_k T(\tau) = f(\tau + 1).$$

The basic fact here (cf. [KM]) is:

If the T-action on W is a single Jordan block[1] with eigenvalue $\lambda = e^{2\pi i \mu}$:

$$\begin{pmatrix} \lambda & 0 & \dots & 0 \\ \lambda & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & \lambda & \lambda \end{pmatrix},$$

the corresponding basis of W takes the form $(g_0(\tau), \dots, g_m(\tau))$ where

$$g_i(\tau) = \sum_{j=0}^{i} \binom{\tau}{j} h_{i-j}(\tau), \quad h_j(\tau) = \sum_{n \in \mathbb{Z}} a_{jn} q^{n+\mu},$$

$$\binom{\tau}{j} = \frac{\tau(\tau-1)\dots(\tau-j+1)}{j!}.$$

If W is 1-dimensional, this reduces to the well-known result:

$$f \in \mathfrak{F},\ f(\tau+1) = f(\tau) \Rightarrow f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n.$$

The more general situation can be reduced to this case.

b) An unrestricted (logarithmic) vector-valued modular form (V, φ) of weight k is a finite-dimensional right Γ-module V together with a morphism

$$\varphi : V \to \mathfrak{F}_k.$$

By a), there is a choice of basis v_i of V so that

$$\varphi(v_i) = f_i(\tau) = \sum_{j=0}^{r_i} \binom{\tau}{j} h_{i,r_i-j}(\tau), \quad h_{ij}(\tau) = \sum_{n\in\mathbb{Z}} a_{ijn} q^{n+\mu_{r_i}},$$

i.e.

$$F(\tau) = \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_p(\tau) \end{pmatrix} = \begin{pmatrix} \vdots \\ h_{i0}(\tau) \\ h_{i1}(\tau) + \tau h_{i0}(\tau) \\ \vdots \\ h_{ir_i}(\tau) + \tau h_{ir_{i-1}}(\tau) + \ldots + \binom{\tau}{r_i} h_{i0}(\tau) \\ \vdots \end{pmatrix}.$$

This is exactly what happens in logarithmic field theory. Here, it can also be shown ([M]) that the exponents μ_i lie in $\mathbb{Q}$, i.e the eigenvalues λ of the T-action are roots of unity. Moreover, the q-expansions $h_{ij}(\tau)$ have at worst poles at infinity, i.e. $a_{ijn} = 0$ for $n \ll 0$. The h_{ij} are trace functions associated to V-modules.

8 The spaces $\mathcal{H}(\rho)$

a) A logarithmic vector-valued modular form (V, φ) is called *holomorphic* if all of the associated q-expansions (i.e. the $h_{ij}(\tau)$) are *holomorphic at infinity* in the sense that no negative powers of q occur.

Given a *meromorphic* vector-valued modular form (e.g., arising from a logarithmic field theory) one can get a holomorphic vector-valued modular form simply by multiplying by some power of the discriminant $\Delta(\tau)$, for example. So for many purposes it is sufficient to deal with holomorphic vector-valued forms.

We can reformulate the idea of a vector-valued modular form of weight k as a pair $(F(\tau), \rho)$ where $F(\tau)$ is a vector of functions in $\mathfrak{F}$ as above, $\rho : \Gamma \to GL_p(\mathbb{C})$ is a representation, and

$$\rho(\gamma)F(\tau) = F|_k\gamma(\tau).$$

b) Let

$$\begin{aligned} \mathcal{H}(k,\rho) &= \{\text{holomorphic vv modular forms of weight } k\}, \\ \mathcal{H}(\rho) &= \{\text{space of all holomorphic vv modular forms}\}, \\ &= \bigoplus_k \mathcal{H}(k,\rho). \end{aligned}$$

Let $\mathfrak{M}$ be the algebra of classical holomorphic modular forms on Γ, so that

$$\mathfrak{M} = \mathbb{C}[E_4, E_6].$$

Pointwise multiplication by $\mathfrak{M}$ turns $\mathcal{H}(\rho)$ into a left $\mathcal{M}$-module:

$$\begin{gathered} \mathcal{M} \times \mathcal{H}(\rho) \to \mathcal{H}(\rho) \\ (g(\tau), F(\tau)) \mapsto g(\tau)F(\tau). \end{gathered}$$

We now have ([MM], [KM])

Theorem: Suppose that the exponents μ are *real*, i.e. the eigenvalues of $\rho(T)$ have absolute value 1. Then $\mathcal{H}(\rho)$ is a free $\mathcal{M}$-module of rank exactly p.

Note that the restriction on the exponents holds in the logarithmic field theory case by Miyamoto's work. The Theorem says that there are p holomorphic logarithmic vector-valued modular forms $F_1(\tau), \ldots, F_p(\tau)$ such that *every* holomorphic logarithmic vector-valued modular form $F(\tau)$ has a unique expression in the form

$$F(\tau) = g_1(\tau)F_1(\tau) + \ldots g_p(\tau)F_p(\tau), \quad g_i(\tau) \in \mathfrak{M}.$$

The proof uses the construction of logarithmic meromorphic vector-valued Poincaré series together with techniques from commutative algebra. The rational case, in which powers of τ are absent, has also been treated by Peter Bantay and Terry Gannon by different methods ([BG]).

9 Some Examples

a) One can try to use the free module theorem to understand all meromorphic vector-valued modular forms (logarithmic or rational). This includes, of course, the vector-valued modular forms arising from logarithmic field theory. One generally needs information about the generators $F_i(\tau)$, e.g. their *weights* as vector-valued modular forms (which are uniquely determined). Bantay and Gannon have some results in this direction ([BG])

b) Here are some examples where we get a complete answer.

Let ρ be the *canonical* 2-dimensional representation of Γ. I.e.

$$\rho(T) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(S) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$F_0(\tau) = \begin{pmatrix} \tau \\ 1 \end{pmatrix} \in \mathcal{H}(-1, \rho).$$

More generally,

$$F_0(\tau) = \begin{pmatrix} \tau^p \\ \vdots \\ \tau \\ 1 \end{pmatrix} \in \mathcal{H}(-p, S^p(\rho)).$$

$S^p(\rho)$ is the pth. symmetric power of ρ. It is an irreducible representation of Γ in which T is represented by a single Jordan block. $F_0(\tau)$ is highly degenerate in that the only q-expansions that occur are constant.

The *modular derivative* $D = D_k$ is the map

$$D : \mathcal{H}(k, \rho) \to \mathcal{H}(k+2, \rho)$$
$$F(\tau) \mapsto \theta F(\tau) + kE_2 F(\tau)$$

where $\theta = qd/dq = (1/2\pi i)d/d\tau$ and E_2 is the weight 2 Eisenstein series.

Using the free module theorem, one can show

Theorem: $F_0, DF_0, D^2F_0, \ldots D^pF_0$ is a free $\mathfrak{M}$-basis for $\mathcal{H}(S^p(\rho))$.

E.g. in the case of ρ this says that

$$\mathcal{H}(\rho) = \mathcal{M}F_0 \oplus \mathcal{M}D_{-1}F_0,$$

$$\mathcal{H}(k-1, \rho) = \left\{ \begin{pmatrix} f - E_2 g \\ \tau(f - E_2 g) + g/2\pi i \end{pmatrix} \mid f \in \mathcal{M}_k, g \in \mathcal{M}_{k-2} \right\}.$$

Thus we see that the q-expansions involved here are (up to scalars) g and $f - E_2 g$ where f, g are modular forms. These are *quasimodular forms of depth at most* 1. More generally, one sees that if $F(\tau) \in \mathcal{H}(k, S^p(\rho))$ then the component functions are *quasimodular forms of depth at most* p.

This suggests the following

Question: Are the q-expansions associated to logarithmic field theories quasimodular forms?

This fits nicely with the idea that in the rational case, the q-expansions are modular forms.

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