Math. Ann.
DOI 10.1007/s00208-009-0385-x

Mathematische Annalen

A generalization of Cachazo–Douglas–Seiberg–Witten conjecture for symmetric spaces

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Received: 2 June 2008 / Revised: 26 June 2008

Abstract We extend the original Cachazo–Douglas–Seiberg–Witten conjecture on the structure of the chiral ring of classical supersymmetric Yang-Mills theory to symmetric spaces.

1 Introduction

Let $\mathfrak{g}$ be a (finite-dimensional) semisimple Lie algebra over the complex numbers $\mathbb{C}$ and let σ be an involution (i.e., an automorphism of order 2) of $\mathfrak{g}$. Let $\mathfrak{k}$ (resp. $\mathfrak{p}$) be the +1 (resp. -1) eigenspace of σ. Then, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a $\mathfrak{k}$-module under the adjoint action. In this paper we only consider those involutions σ such that $\mathfrak{p}$ is an irreducible $\mathfrak{k}$-module.

We fix a $\mathfrak{g}$-invariant nondegenerate (symmetric) bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Then, the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is an orthogonal decomposition.

Let $R := \wedge(\mathfrak{p} \oplus \mathfrak{p})$ be the exterior algebra on two copies of $\mathfrak{p}$. To distinguish, we denote the first copy of $\mathfrak{p}$ by $\mathfrak{p}_1$ and the second copy by $\mathfrak{p}_2$. It is bigraded by declaring $\mathfrak{p}_1$ (resp. $\mathfrak{p}_2$) to have bidegree (1,0) (resp. (0,1)). Choose any basis $\{e_i\}$ of $\mathfrak{p}$ and let $\{f_i\}$ be the dual basis of $\mathfrak{p}$, i.e.,

$$\langle e_i, f_j \rangle = \delta_{i,j}.$$

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Define a $\mathfrak{k}$-module map (under the adjoint action)

$$c_3 : \mathfrak{k} \to \mathfrak{p} \otimes \mathfrak{p}, \quad c_3(x) = \sum_i [x, e_i] \otimes f_i.$$

(Observe that $\mathfrak{p} \otimes \mathfrak{p}$ is viewed as a subspace of R as $\mathfrak{p}_1 \otimes \mathfrak{p}_2$.)

It is easy to see that c_3 does not depend upon the choice of the basis $\{e_i\}$. Projected onto $\wedge^2(\mathfrak{p})$, we get a $\mathfrak{k}$-module map $\mathfrak{k} \to \wedge^2(\mathfrak{p})$. This map is denoted by c_1 considered as a map $\mathfrak{k} \to \wedge^2(\mathfrak{p}_1)$, and similarly for $c_2 : \mathfrak{k} \to \wedge^2(\mathfrak{p}_2)$. We denote the image of c_i by C_i. Let J be the (bigraded) ideal of R generated by $C_1 \oplus C_2 \oplus C_3$ and let us consider the quotient algebra

$$A := R/J.$$

The algebra A is a $\mathfrak{k}$-algebra (induced from the adjoint action of $\mathfrak{k}$) and let $A^{\mathfrak{k}}$ be the subalgebra of $\mathfrak{k}$-invariants. The algebra $A^{\mathfrak{k}}$ contains the element $S := \sum e_i \otimes f_i$ in bidegree (1,1).

The aim of this paper is to understand the structure of the algebra $A^{\mathfrak{k}}$.

In the case when $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}$ for a simple Lie algebra $\mathfrak{s}$ and σ is the involution which switches the two factors, the study of the structure of $A^{\mathfrak{k}}$ was initiated by Cachazo–Douglas–Seiberg–Witten who made the following conjecture. (Observe that in this case $\mathfrak{k}$ and $\mathfrak{p}$ both can be identified with $\mathfrak{s}$ and the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$ under this identification is nothing but the adjoint action of $\mathfrak{s}$ on itself.) We will refer to this as the *diagonal case*.

Conjecture 1.1 [2] (i) *The subalgebra* $A^{\mathfrak{k}}$ *of* $\mathfrak{k}$*-invariants in* A *is generated, as an algebra, by the element* S.

(ii) $S^h = 0$.
(iii) $S^{h-1} \neq 0$,

where h *is the dual Coxeter number of* $\mathfrak{k} = \mathfrak{s}$.

They proved the conjecture for $\mathfrak{s} = sl_N$ in [2], and Witten proved it for $\mathfrak{s} = sp_N$ in [16]. He also proved parts (i) and (ii) of the conjecture for $\mathfrak{s} = so_N$ in [16]. Subsequently, Etingof–Kac proved the conjecture for $\mathfrak{s}$ of type G_2 by using the theory of abelian ideals and Etingof proved it for any classical $\mathfrak{s}$. Kumar proved part (i) of the conjecture uniformly in [13] using geometric and topological methods.

Returning to the general case of any involution σ, we prove the following analogous result (cf. Theorem 4.8) which is the main result of this paper.

Theorem 1.2 *Let* σ *be any involution of a simple Lie algebra* $\mathfrak{g}$ *such that* $\mathfrak{p}$ *is an irreducible module under the adjoint action of* $\mathfrak{k}$*. Then, the subalgebra* $A^{\mathfrak{k}}$ *of* $\mathfrak{k}$*-invariants in* A *is generated, as an algebra, by the element* S.

Analogous to our proof in the diagonal case, we need to consider the algebra $B := R/\langle C_1 \oplus C_2 \rangle$. We show (cf. Theorem 3.1) that the subalgebra $B^{\mathfrak{k}}$ of $\mathfrak{k}$-invariants of B is graded isomorphic with the singular cohomology with complex coefficients

$H^*(\mathcal{Y})$ of a certain finite-dimensional projective subvariety $\mathcal{Y}$ of the twisted affine Grassmannian $\mathcal{X}_\sigma$ (cf. Sect. 2 for the definitions of $\mathcal{X}_\sigma$ and $\mathcal{Y}$). The definition of the subvariety $\mathcal{Y}$ is motivated from the theory of abelian subspaces of $\mathfrak{p}$. The main ingredients in our proof of Theorem 3.1 are: result of Garland–Lepowsky on the Lie algebra cohomology of the nil-radical $\hat{\mathfrak{u}}_\sigma$ of a maximal parabolic subalgebra of twisted affine Kac–Moody Lie algebras; the 'diagonal' cohomology of $\hat{\mathfrak{u}}_\sigma$ introduced by Kostant; certain results of Han and Cellini–Frajria–Papi on abelian subspaces of $\mathfrak{p}$; and a certain deformation of the singular cohomology of $\mathcal{X}_\sigma$ introduced by Belkale–Kumar.

Having identified the algebra $B^{\mathfrak{k}}$ with $H^*(\mathcal{Y})$, we next use the fact that $H^*(\mathcal{X}_\sigma)$ surjects onto $H^*(\mathcal{Y})$ under the restriction map. We study the cohomology algebra $H^*(\mathcal{X}_\sigma)$ in Sect. 4. The results here are more involved than in the diagonal case. One mazor difficulty arises from the fact that the fibration

$$\Omega_1^\sigma(G_o) \to \Omega^\sigma(G_o)/K_o \xrightarrow{\gamma} G_o/K_o$$

is nontrivial (cf. Sect. 4 for various notation). To complete the proof of our Theorem 4.8, we show that all but one of the algebra generators of $H^*(\mathcal{X}_\sigma)$ go to zero under the canonical projection map $B^{\mathfrak{k}} \to A^{\mathfrak{k}}$ and the remaining one generator goes to S.

Finally, analogous to the Cachazo–Douglas–Seiberg–Witten Conjecture, we make the following conjecture under the assumption of Theorem 1.2.

Conjecture 1.3 *$S^{h+1} = 0$ and $S^h \neq 0$ in $A^{\mathfrak{k}}$, where $h = h_{\mathfrak{g}} - h_{\mathfrak{k}}$ ($h_{\mathfrak{g}}$ being the dual Coxeter number of $\mathfrak{g}$).*

It is easy to verify that the above conjecture is true in the case $\mathfrak{g}$ is the Lie algebra so_{2n} and σ is the involution of $\mathfrak{g}$ such that $\mathfrak{k} = so_{2n-1}$. In this case $h = 1$.

Unless otherwise stated, by the cohomology $H^*(X)$ of a topological space X we mean the singular cohomology $H^*(X, \mathbb{C})$ with complex coefficients.

2 Preliminaries and notation

2.1 Twisted affine Lie algebras

Let $\mathfrak{g}$ be a (finite-dimensional) simple Lie algebra over $\mathbb{C}$ and let σ be an involution of $\mathfrak{g}$. Let $\mathfrak{k} \subset \mathfrak{g}$ be the $+1$ eigenspace of σ (which is a reductive subalgebra of $\mathfrak{g}$) and let $\mathfrak{p}$ be the -1 eigenspace of σ, which is a $\mathfrak{k}$-module under the adjoint action. As in the introduction, we only consider those involutions σ such that $\mathfrak{p}$ is an irreducible $\mathfrak{k}$-module. *This will be our tacit assumption on σ throughout the paper.*

Fix a Cartan subalgebra $\mathfrak{h}_\sigma$ and a Borel subalgebra $\mathfrak{b}_\sigma \supset \mathfrak{h}_\sigma$ of $\mathfrak{k}$. Let $\mathfrak{n}_\sigma$ be the nil-radical of $\mathfrak{b}_\sigma$. Associated to the pair $(\mathfrak{g}, \sigma)$ we have the *twisted affine Kac–Moody Lie algebra*

$$\hat{\mathfrak{g}}_\sigma := \left(\sum_{i \in \mathbb{Z}} \mathfrak{g}_i \otimes t^i \right) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathfrak{g}_{2i} := \mathfrak{k}$ and $\mathfrak{g}_{2i+1} := \mathfrak{p}$ for any $i \in \mathbb{Z}$. The bracket in $\hat{\mathfrak{g}}_\sigma$ is defined as follows:

$$\begin{aligned}&\left[x \otimes t^m + \lambda c + \mu d, x' \otimes t^{m'} + \lambda' c + \mu' d\right]\\ &\quad = \left([x, x'] \otimes t^{m+m'} + \mu m' x' \otimes t^{m'} - \mu' m x \otimes t^m\right) + m\, \delta_{m,-m'} \langle x, x' \rangle c,\end{aligned}$$

where $\langle\ ,\ \rangle$ is the normalized $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$, normalized so that the induced form on $\mathfrak{h}^*$ satisfies $\langle \theta, \theta \rangle = 2$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra and $\theta \in \mathfrak{h}^*$ is the highest root of $\mathfrak{g}$ (with respect to any choice of positive roots).

The Lie algebra $\hat{\mathfrak{g}}_\sigma$ is a subalgebra of the affine Kac–Moody algebra

$$\hat{\mathfrak{g}} := \left(\sum_{i \in \mathbb{Z}} \mathfrak{g} \otimes t^i\right) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with the bracket defined by the same formula as above.

We define the following subalgebras of $\hat{\mathfrak{g}}_\sigma$ called the *standard Cartan*, *standard Borel* and the *standard maximal parabolic subalgebra* respectively:

$$\begin{aligned}\hat{\mathfrak{h}}_\sigma &:= \mathfrak{h}_\sigma \otimes t^0 \oplus \mathbb{C}c \oplus \mathbb{C}d,\\ \hat{\mathfrak{b}}_\sigma &:= \mathfrak{b}_\sigma \otimes t^0 \oplus \left(\sum_{i>0} \mathfrak{g}_i \otimes t^i\right) \oplus \mathbb{C}c \oplus \mathbb{C}d, \text{ and}\\ \hat{\mathfrak{p}}_\sigma &:= \left(\sum_{i \geq 0} \mathfrak{g}_i \otimes t^i\right) \oplus \mathbb{C}c \oplus \mathbb{C}d.\end{aligned}$$

We also have the *nil-radicals* $\hat{\mathfrak{n}}_\sigma$ of $\hat{\mathfrak{b}}_\sigma$ and $\hat{\mathfrak{u}}_\sigma$ of $\hat{\mathfrak{p}}_\sigma$ and the Levi subalgebra $\hat{\mathfrak{r}}_\sigma$ of $\hat{\mathfrak{p}}_\sigma$ defined as follows:

$$\begin{aligned}\hat{\mathfrak{n}}_\sigma &:= \mathfrak{n}_\sigma \otimes t^0 \oplus \left(\sum_{i>0} \mathfrak{g}_i \otimes t^i\right),\\ \hat{\mathfrak{u}}_\sigma &:= \sum_{i>0} \mathfrak{g}_i \otimes t^i, \text{ and}\\ \hat{\mathfrak{r}}_\sigma &:= \mathfrak{k} \otimes t^0 \oplus \mathbb{C}c \oplus \mathbb{C}d.\end{aligned}$$

The evaluation at 1 gives rise to a Lie algebra homomorphism

$$ev_1 : \hat{\mathfrak{g}}_\sigma \to \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c and d are central in the right side.

Associated to the twisted affine Kac–Moody Lie algebra $\hat{\mathfrak{g}}_\sigma$ and its subalgebras $\hat{\mathfrak{p}}_\sigma$ and $\hat{\mathfrak{b}}_\sigma$, we have the twisted affine Kac–Moody group $\mathcal{G}_\sigma$, the standard maximal parabolic subgroup $\mathcal{P}_\sigma$ and the standard Borel subgroup $\mathcal{B}_\sigma$ respectively (cf. [12, Chapter 6]).

Let W_σ be the (finite) Weyl group of $(\mathfrak{k}, \mathfrak{h}_\sigma)$ and let $\mathcal{W}_\sigma$ be the (affine) Weyl group of $\left(\hat{\mathfrak{g}}_\sigma, \hat{\mathfrak{h}}_\sigma\right)$. Let $\hat{\Delta}_\sigma^+ \subset (\hat{\mathfrak{h}}_\sigma)^*$ be the set of positive roots of $\hat{\mathfrak{g}}_\sigma$, i.e., the set of roots for the subalgebra $\hat{\mathfrak{n}}_\sigma$ with respect to the adjoint action of $\hat{\mathfrak{h}}_\sigma$. We set $\hat{\Delta}_\sigma^- = -\hat{\Delta}_\sigma^+$. For any $w \in \mathcal{W}_\sigma$, define

$$\Phi(w) := \hat{\Delta}_\sigma^+ \cap w\, \hat{\Delta}_\sigma^-, \text{ and}$$
$$\hat{\mathfrak{n}}_\sigma(w) := \bigoplus_{\alpha \in \Phi(w)} \left(\hat{\mathfrak{g}}_\sigma\right)_\alpha,$$

where $\left(\hat{\mathfrak{g}}_\sigma\right)_\alpha$ denotes the root space of $\hat{\mathfrak{g}}_\sigma$ corresponding to the root α. Since each root in $\Phi(w)$ is real, $\left(\hat{\mathfrak{g}}_\sigma\right)_\alpha$ is one-dimensional for each $\alpha \in \Phi(w)$.

2.2 Abelian subspaces of $\mathfrak{p}$

Let $\mathcal{W}'_\sigma \subset \mathcal{W}_\sigma$ be the set of minimal coset representatives in the cosets $\mathcal{W}_\sigma / W_\sigma$.

Following [4], we call an element $w \in \mathcal{W}_\sigma$ *minuscule* if

$$\hat{\mathfrak{n}}_\sigma(w^{-1}) \subset \mathfrak{p} \otimes t.$$

Let us denote the set of minuscule elements in $\mathcal{W}_\sigma$ by $\mathcal{W}_\sigma^{\text{minu}}$. Then, it is easy to see that $\mathcal{W}_\sigma^{\text{minu}} \subset \mathcal{W}'_\sigma$ and, clearly, it is a finite set.

We recall the following result from [4, Theorem 3.2] (also see [6]).

Theorem 2.3 *There is a bijection between* $\mathcal{W}_\sigma^{\text{minu}}$ *and the set* Ξ *of* $\mathfrak{b}_\sigma$*-stable abelian subspaces of* $\mathfrak{p}$ *given by* $w \mapsto ev_1(\hat{\mathfrak{n}}_\sigma(w^{-1}))$. *In particular, the cardinality* $|\mathcal{W}_\sigma^{\text{minu}}| = |\Xi|$.

We recall the Bruhat decomposition (cf. [12, Corollary 6.1.20]) of the projective ind-variety

$$\mathcal{X}_\sigma := \mathcal{G}_\sigma / \mathcal{P}_\sigma = \bigsqcup_{w \in \mathcal{W}'_\sigma} \mathcal{B}_\sigma w \mathcal{P}_\sigma / \mathcal{P}_\sigma,$$

where the *Bruhat cell* $C(w) := \mathcal{B}_\sigma w \mathcal{P}_\sigma / \mathcal{P}_\sigma$ is isomorphic to the affine space $\mathbb{C}^{\ell(w)}$ ($\ell(w)$ being the length of w in the Coxeter group $\mathcal{W}_\sigma$). Moreover, for any $w \in \mathcal{W}'_\sigma$, the Zariski closure

$$\overline{C(w)} = \bigsqcup_{\substack{v \in \mathcal{W}'_\sigma \text{ and} \\ v \leq w}} C(v).$$

Define a subset $\mathcal{Y}$ of $\mathcal{G}_\sigma / \mathcal{P}_\sigma$ by

$$\mathcal{Y} = \bigsqcup_{w \in \mathcal{W}_\sigma^{\text{minu}}} C(w).$$

Then, $\mathcal{Y}$ is a (finite-dimensional) projective subvariety of $\mathcal{G}_\sigma/\mathcal{P}_\sigma$. This follows from the following.

Lemma 2.4 *For $w \in \mathcal{W}_\sigma^{\mathrm{minu}}$ and any $u \in \mathcal{W}'_\sigma$ such that $u \leq w$, we have $u \in \mathcal{W}_\sigma^{\mathrm{minu}}$.*

Proof (due to P. Frajria and P. Papi) By the definition, an element $u \in \mathcal{W}_\sigma$ is minuscule iff $\beta(d) = 1$ for all $\beta \in \Phi(u^{-1})$. By the L-shellability of the Bruhat order in $\mathcal{W}'_\sigma$, we can assume that $w = us_\alpha$, where $\alpha \in \hat{\Delta}_\sigma^+$ is a real root and s_α is the reflection through α: $s_\alpha\lambda = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha$ for $\lambda \in (\hat{\mathfrak{h}}_\sigma)^*$. Since $u < w$, we have $w\alpha \in \hat{\Delta}_\sigma^-$, and hence $\alpha \in \Phi(w^{-1})$. In particular, $\alpha(d) = 1$. Since $u \in \mathcal{W}'_\sigma$, we have $\beta(d) \neq 0$ for any $\beta \in \Phi(u^{-1})$. Thus, it suffices to prove that for any $\beta \in \hat{\Delta}_\sigma^+$ such that $\beta(d) > 1$, we have $u\beta \in \hat{\Delta}_\sigma^+$. Observe that since $\beta(d) > 1$, $w\beta \in \hat{\Delta}_\sigma^+$.

There are three cases to consider:

Case I: $s_\alpha\beta \in \hat{\Delta}_\sigma^-$.
In this case, $\langle\beta, \alpha^\vee\rangle > 0$. Thus,

$$u\beta = w(s_\alpha\beta) = w(\beta - \langle\beta, \alpha^\vee\rangle\alpha) = w\beta - \langle\beta, \alpha^\vee\rangle w\alpha \in \hat{\Delta}_\sigma^+,$$

since $w\alpha \in \hat{\Delta}_\sigma^-$.

Case II: $s_\alpha\beta \in \hat{\Delta}_\sigma^+$ and $s_\alpha\beta(d) \neq 1$.
In this case, $s_\alpha\beta \notin \Phi(w^{-1})$, i.e., $u\beta = ws_\alpha\beta \in \hat{\Delta}_\sigma^+$.

Case III: $s_\alpha\beta \in \hat{\Delta}_\sigma^+$ and $s_\alpha\beta(d) = 1$.
In this case,

$$\begin{aligned} s_\alpha\beta(d) &= \beta(d) - \langle\beta, \alpha^\vee\rangle\,\alpha(d) \\ &= \beta(d) - \langle\beta, \alpha^\vee\rangle = 1, \ \text{ since } \alpha(d) = 1. \end{aligned}$$

Thus, $\langle\beta, \alpha^\vee\rangle = \beta(d) - 1 > 0$ (since $\beta(d) > 1$) and hence $u\beta = ws_\alpha\beta = w\beta - \langle\beta, \alpha^\vee\rangle(w\alpha) \in \hat{\Delta}_\sigma^+$, since $w\alpha \in \hat{\Delta}_\sigma^-$. This proves the lemma. □

3 Topological identification of the algebra $B^{\mathfrak{k}}$

Consider the $\mathbb{Z}_+$-graded $\mathfrak{k}$-algebra

$$B := \frac{\wedge(\mathfrak{p}) \otimes \wedge(\mathfrak{p})}{\langle C_1 \oplus C_2\rangle},$$

where C_1 and C_2 are defined in the Introduction.

Following is the first main result of this paper.

Theorem 3.1 *The singular cohomology $H^*(\mathcal{Y}, \mathbb{C})$ of $\mathcal{Y}$ with complex coefficients is isomorphic as a $\mathbb{Z}_+$-graded algebra with the graded algebra of $\mathfrak{k}$-invariants $B^{\mathfrak{k}}$.*

Before we come to the proof of the theorem, we need to recall the following results. The first theorem is a special case of a result due to Garland–Lepowsky and the second theorem is due to Han.

Theorem 3.2 [12, Theorem 3.2.7 and Identity (3.2.11.3)] *As a module for* $\hat{\mathfrak{r}}_\sigma$*,*

$$H^p(\hat{\mathfrak{u}}_\sigma, \mathbb{C}) \simeq \bigoplus_{\substack{w \in \mathcal{W}'_\sigma \\ \ell(w)=p}} L(w^{-1}\hat{\rho} - \hat{\rho}),$$

where $\hat{\rho}$ *is any element of* $\left(\hat{\mathfrak{h}}_\sigma\right)^*$ *satisfying* $\hat{\rho}(\alpha_i^\vee) = 1$ *for all the simple coroots* $\{\alpha_0^\vee, \dots, \alpha_\ell^\vee\} \subset \hat{\mathfrak{h}}_\sigma$ *of* $\hat{\mathfrak{g}}_\sigma$ *and* $L(w^{-1}\hat{\rho} - \hat{\rho})$ *denotes the irreducible* $\hat{\mathfrak{r}}_\sigma$*-module with highest weight* $w^{-1}\hat{\rho} - \hat{\rho}$*. Similarly, by* [12, Theorem 3.2.7]*,*

$$H^p(\hat{\mathfrak{u}}_\sigma^-, \mathbb{C}) \simeq \bigoplus_{\substack{w \in \mathcal{W}'_\sigma, \\ \ell(w)=p}} L(w^{-1}\hat{\rho} - \hat{\rho})^*,$$

where $\hat{\mathfrak{u}}_\sigma^- := \sum_{i<0} \mathfrak{g}_i \otimes t^i$*.*

For any $\mathfrak{b}_\sigma$-stable abelian subspace $I \subset \mathfrak{p}$ of dimension n, $\wedge^n(I)$ is a $\mathfrak{b}_\sigma$-stable line in $\wedge^n(\mathfrak{p})$ and hence generates an irreducible $\mathfrak{k}$-submodule V_I of $\wedge^n(\mathfrak{p})$ with highest weight space $\wedge^n(I)$. Thus, we get a $\mathfrak{k}$-module map

$$\bigoplus_{I \in \Xi} V_I \to \wedge(\mathfrak{p}) \to \wedge(\mathfrak{p})/\langle C_1 \rangle.$$

If I corresponds via Theorem 2.3 to the element $w \in \mathcal{W}_\sigma^{\text{minu}}$, then V_I has highest weight $(w^{-1}\hat{\rho} - \hat{\rho})_{|\mathfrak{h}_\sigma}$.

Theorem 3.3 [9, Theorem (4.7)] *The above* $\mathfrak{k}$*-module map*

$$\bigoplus_{I \in \Xi} V_I \to \wedge(\mathfrak{p})/\langle C_1 \rangle$$

is an isomorphism. Moreover, by [14, Theorem 4.13(2)]*, the* $\mathfrak{k}$*-module* $\bigoplus_{I \in \Xi} V_I$ *is multiplicity free.*

For any $w \in \mathcal{W}'_\sigma$, define the Schubert cohomology class $\varepsilon^w \in H^{2\ell(w)}(\mathcal{X}_\sigma, \mathbb{Z})$ by

$$\varepsilon^w\left([\overline{C(u)}]\right) = \delta_{w,u} \text{ for } u \in \mathcal{W}'_\sigma,$$

where $[\overline{C(u)}] \in H_{2\ell(u)}(\mathcal{X}_\sigma, \mathbb{Z})$ denotes the fundamental homology class of $\overline{C(u)}$.

Following Belkale and Kumar [1, §6], we define a new product $\odot_0$ in $H^*(\mathcal{X}_\sigma, \mathbb{Z})$ as follows. Express the standard cup product

$$\varepsilon^u \cdot \varepsilon^v = \sum_{w \in \mathcal{W}'_\sigma} c_{u,v}^w \varepsilon^w.$$

Now, define

$$\varepsilon^u \odot_0 \varepsilon^v = \sum c_{u,v}^w \, \delta_{d_{u,v}^w, 0} \, \varepsilon^w,$$

where

$$d^w_{u,v} := \left(u^{-1}\hat{\rho} + v^{-1}\hat{\rho} - w^{-1}\hat{\rho} - \hat{\rho}\right)(d).$$

The product $\odot_0$ descends to a product in $H^*(\mathcal{Y}, \mathbb{Z})$ under the restriction map $H^*(\mathcal{X}_\sigma, \mathbb{Z}) \to H^*(\mathcal{Y}, \mathbb{Z})$.

Lemma 3.4 *The product $\odot_0$ coincides with the standard cup product in $H^*(\mathcal{Y}, \mathbb{Z})$.*

Proof For any $w \in \mathcal{W}_\sigma$, by [12, Corollary 1.3.22],

$$|\Phi(w)| = \hat{\rho} - w\hat{\rho},$$

where

$$|\Phi(w)| := \sum_{\beta \in \Phi(w)} \beta.$$

Thus, for any $w \in \mathcal{W}^{\text{minu}}_\sigma$, by its definition,

(1) $$(\hat{\rho} - w^{-1}\hat{\rho})(d) = \ell(w).$$

To prove the lemma, it suffices to show that whenever $c^w_{u,v} \neq 0$ for $u, v, w \in \mathcal{W}^{\text{minu}}_\sigma$, $d^w_{u,v} = 0$. But, $c^w_{u,v} \neq 0$ gives

(2) $$\ell(w) = \ell(u) + \ell(v).$$

Thus,

$$\begin{aligned} d^w_{u,v} &= \left(u^{-1}\hat{\rho} - \hat{\rho} + v^{-1}\hat{\rho} - \hat{\rho} - \left(w^{-1}\hat{\rho} - \hat{\rho}\right)\right)(d) \\ &= -\ell(u) - \ell(v) + \ell(w) \quad \text{by (1)} \\ &= 0 \quad \text{by (2)}. \end{aligned}$$

□

Proof of Theorem 3.1 The cohomology modules $H^p(\hat{\mathfrak{u}}_\sigma)$ and $H^p(\hat{\mathfrak{u}}^-_\sigma)$ acquire a grading coming from the total degree of t in $\wedge^p(\hat{\mathfrak{u}}_\sigma)$ and $\wedge^p(\hat{\mathfrak{u}}^-_\sigma)$ respectively. This decomposes

$$H^p(\hat{\mathfrak{u}}_\sigma) = \bigoplus_{m \in \mathbb{Z}_+} H^p_{(-m)}(\hat{\mathfrak{u}}_\sigma),$$

where $H^p_{(-m)}(\hat{\mathfrak{u}}_\sigma)$ denotes the space of elements of $H^p(\hat{\mathfrak{u}}_\sigma)$ of total t-degree $-m$. Define the diagonal cohomology

$$H^*_D(\hat{\mathfrak{u}}_\sigma) := \bigoplus_{p \in \mathbb{Z}_+} H^p_{(-p)}(\hat{\mathfrak{u}}_\sigma),$$

which is a subalgebra of $H^*(\hat{\mathfrak{u}}_\sigma)$, and similarly define $H^*_D(\hat{\mathfrak{u}}^-_\sigma)$.

Let $\phi : \wedge^p(\mathfrak{p}) \to H^p_{(-p)}(\hat{\mathfrak{u}}_\sigma)$ be the map induced from the map $\bar{\phi} : \wedge^p(\mathfrak{p}) \to C^p_{(-p)}(\hat{\mathfrak{u}}_\sigma)$,

$$\bar{\phi}\left(x_1 \wedge \cdots \wedge x_p\right)\left(y_1 \otimes t \wedge \cdots \wedge y_p \otimes t\right) = \det\left(\langle x_i, y_j\rangle\right)_{i,j},$$

(for $x_i, y_j \in \mathfrak{p}$) by taking the cohomology class of the image, where $C^p_{(-p)}(\hat{\mathfrak{u}}_\sigma)$ denotes the space of p-cochains on $\hat{\mathfrak{u}}_\sigma$ with total t-degree $-p$. Clearly, $\bar{\phi}(x_1 \wedge \cdots \wedge x_p)$ is a cocycle and, moreover, $\bar{\phi}$ (and hence ϕ) is surjective. It is easy to see that $\operatorname{Ker}\left(\phi_{|_{\wedge^2(\mathfrak{p})}}\right) = C_1$. Now, take any $\omega \in C^{p-1}_{(-p)}(\hat{\mathfrak{u}}_\sigma)$. We can write

$$\omega = \sum_{i=1}^{N} \omega_1^i \wedge \omega_2^i,$$

for some $\omega_1^i \in C^1_{(-2)}(\hat{\mathfrak{u}}_\sigma)$ and $\omega_2^i \in C^{p-2}_{(-p+2)}(\hat{\mathfrak{u}}_\sigma)$. Then,

$$\delta\omega = \sum_{i=1}^{N} (\delta\omega_1^i) \wedge \omega_2^i,$$

since ω_2^i are δ-closed, where δ is the standard differential of the cochain complex $C^*(\hat{\mathfrak{u}}_\sigma)$.

From this it is easy to see that $\operatorname{Ker}\phi = \langle C_1\rangle$. Thus, we get a graded algebra isomorphism commuting with the $\mathfrak{k}$-module structures:

$$\frac{\wedge^*(\mathfrak{p})}{\langle C_1\rangle} \simeq H^*_D(\hat{\mathfrak{u}}_\sigma). \tag{1}$$

In exactly the same way, we get an isomorphism of graded algebras commuting with the $\mathfrak{k}$-module structures:

$$\frac{\wedge^*(\mathfrak{p})}{\langle C_2\rangle} \simeq H^*_D(\hat{\mathfrak{u}}^-_\sigma). \tag{2}$$

In particular, $\frac{\wedge^p(\mathfrak{p})}{\langle C_1\rangle \cap \wedge^p(\mathfrak{p})}$ is a self-dual $\mathfrak{k}$-module for any $p \geq 0$.

Combining (1)–(2), we get an isomorphism (for any $p, q \geq 0$)

$$\left[\frac{\wedge^p(\mathfrak{p})}{\langle C_1\rangle \cap \wedge^p(\mathfrak{p})} \otimes \frac{\wedge^q(\mathfrak{p})}{\langle C_2\rangle \cap \wedge^q(\mathfrak{p})}\right]^{\mathfrak{k}} \simeq \left[H^p_D(\hat{\mathfrak{u}}_\sigma) \otimes H^q_D(\hat{\mathfrak{u}}^-_\sigma)\right]^{\mathfrak{k}}. \tag{3}$$

Since $\frac{\wedge^*(\mathfrak{p})}{\langle C_1\rangle}$ is multiplicity free (by Theorem 3.3) and $\frac{\wedge^p(\mathfrak{p})}{\langle C_1\rangle \cap \wedge^p(\mathfrak{p})}$ is self-dual for any $p \geq 0$, the left side of (3) is nonzero only if $p = q$. Moreover, c acts trivially on $H^p_D(\hat{\mathfrak{u}}_\sigma) \otimes H^q_D(\hat{\mathfrak{u}}^-_\sigma)$ and d acts via the multiplication by $q - p$. Thus, we have a graded algebra isomorphism:

$$\left[\frac{\wedge^*(\mathfrak{p})}{\langle C_1\rangle} \otimes \frac{\wedge^*(\mathfrak{p})}{\langle C_2\rangle}\right]^{\mathfrak{k}} \xrightarrow{\sim} \left[H^*_D(\hat{\mathfrak{u}}_\sigma) \otimes H^*_D(\hat{\mathfrak{u}}^-_\sigma)\right]^{\hat{\mathfrak{t}}_\sigma}. \tag{4}$$

By Theorem 3.2, we get

$$H_D^p(\hat{\mathfrak{u}}_\sigma) \simeq H_D^p(\hat{\mathfrak{u}}_\sigma^-)^* \simeq \bigoplus_{\substack{w \in \mathcal{W}_\sigma^{\text{minu}} \\ \ell(w)=p}} L(w^{-1}\hat{\rho} - \hat{\rho}), \tag{5}$$

as $\hat{\mathfrak{r}}_\sigma$-modules. Combining (4)–(5), we get the isomorphism

$$\left[\frac{\wedge^*(\mathfrak{p})}{\langle C_1 \rangle} \otimes \frac{\wedge^*(\mathfrak{p})}{\langle C_2 \rangle}\right]^{\mathfrak{k}} \simeq \bigoplus_{w \in \mathcal{W}_\sigma^{\text{minu}}} \left[L(w^{-1}\hat{\rho} - \hat{\rho}) \otimes L(w^{-1}\hat{\rho} - \hat{\rho})^*\right]^{\hat{\mathfrak{r}}_\sigma}. \tag{6}$$

Now, by a similar argument to that given in [13, Section 2.4], the proof of Theorem 3.1 follows. We omit the details. □

4 Structure of the algebra $A^{\mathfrak{k}}$

Let G be a connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$. The involution σ of $\mathfrak{g}$, of course, induces an involution of G. Choose a maximal compact subgroup G_o of G which is stable under σ and such that the subgroup $K_o := G_o^\sigma$ of σ-invariants is a maximal compact subgroup of $K := G^\sigma$ (cf. [10, Chapter 6, §2]). Moreover, as is well known, K is connected and hence so is K_o.

Let $\Omega^\sigma(G_o)$ be the space of all continuous maps $f : S^1 \to G_o$ which are σ-equivariant, i.e.,

$$f(-z) = \sigma(f(z)) \quad \text{for all } z \in S^1.$$

We put the compact-open topology on $\Omega^\sigma(G_o)$. Clearly, the subspace of constant loops can be identified with K_o. Equivalently, we can view $\Omega^\sigma(G_o)$ as the space of continuous maps $\bar{f} : [0, 2\pi] \to G_o$ such that

$$\bar{f}(t + \pi) = \sigma(\bar{f}(t)), \quad \text{for all } 0 \le t \le \pi.$$

In particular, $\bar{f}(2\pi) = \sigma^2(\bar{f}(0)) = \bar{f}(0)$. The correspondence $f \rightsquigarrow \bar{f}$ is given by $\bar{f}(t) = f(e^{it})$, for $0 \le t \le 2\pi$.

Consider the fibration

$$\Omega_1^\sigma(G_o) \to \Omega^\sigma(G_o)/K_o \xrightarrow{\gamma} G_o/K_o,$$

where $\gamma(fK_o) = f(1)K_o$ for $f \in \Omega^\sigma(G_o)$ and $\Omega_1^\sigma(G_o)$ is the subspace of $\Omega^\sigma(G_o)$ consisting of those f such that $f(1) = 1$.

Of course, $\Omega_1^\sigma(G_o)$ can be identified with the based loop space $\Omega_1(G_o)$ of G_o under $f \rightsquigarrow \bar{f}_{|[0,\pi]}$.

Define the $\mathfrak{k}$-module map $\bar{c} : \mathfrak{k}^* \to \wedge^2(\mathfrak{p})^*$ by $(\bar{c}f)(x \wedge y) = f([x, y])$, for $x, y \in \mathfrak{p}$. This gives rise to the algebra homomorphism (still denoted by)

$$\bar{c} : S(\mathfrak{k}^*) \to \wedge(\mathfrak{p})^*.$$

Consider the restriction of $\bar{c}$ to the subring of $\mathfrak{k}$-invariants

$$c : S(\mathfrak{k}^*)^{\mathfrak{k}} \to C(\mathfrak{g}, \mathfrak{k}) \simeq [\wedge(\mathfrak{p})^*]^{\mathfrak{k}},$$

where $C(\mathfrak{g}, \mathfrak{k})$ is the standard cochain complex for the Lie algebra pair $(\mathfrak{g}, \mathfrak{k})$.

Then, the map c is the Chern–Weil homomorphism with respect to a G_o-invariant connection on the G_o-equivariant principle K_o-bundle $G_o \to G_o/K_o$.

Observe that since $\mathfrak{k}$ is the $+1$ eigenspace of an involution of $\mathfrak{g}$, the differential $\delta \equiv 0$ on $C^*(\mathfrak{g}, \mathfrak{k})$. Thus,

$$C^*(\mathfrak{g}, \mathfrak{k}) \simeq H^*(\mathfrak{g}, \mathfrak{k}) \simeq H^*(G_o/K_o).$$

Thus, in our case, we can think of c as the map $c : S(\mathfrak{k}^*)^{\mathfrak{k}} \to H^*(\mathfrak{g}, \mathfrak{k}) \simeq H^*(G_o/K_o)$.

We now recall the following result due to H. Cartan on the cohomology of G_o/K_o with complex coefficients (cf. [3, Sect. 10]).

Theorem 4.1 *There exists a finite-dimensional graded subspace $V \subset H^*(G_o/K_o)$ concentrated in odd degrees such that, as graded algebras,*

$$H^*(G_o/K_o) \simeq \wedge(V) \otimes \operatorname{Im} c.$$

Corollary 4.2 *Consider the map $\gamma : \Omega^\sigma(G_o)/K_o \to G_o/K_o$ defined earlier (obtained from the evaluation at 1). Then, the induced map in cohomology*

$$\gamma^* : H^*(G_o/K_o) \to H^*(\Omega^\sigma(G_o)/K_o),$$

under the identification

$$H^*(G_o/K_o) \simeq \wedge(V) \otimes \operatorname{Im} c$$

of the above theorem, satisfies

$$\gamma^*_{|V} \equiv 0.$$

In particular, $\operatorname{Im}(\gamma^*) = \gamma^*(\operatorname{Im} c)$.

Proof This follows immediately from the fact that $H^*(\Omega^\sigma(G_o)/K_o)$ is concentrated in even degrees only and V lies in odd cohomological degrees. □

Let $L^\sigma(\mathfrak{g})$ be the twisted loop algebra $\bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \otimes t^i$, i.e., $L^\sigma(\mathfrak{g})$ is the space of all algebraic maps $f : \mathbb{C}^* \to \mathfrak{g}$ satisfying $f(-z) = \sigma(f(z))$ for all $z \in \mathbb{C}^*$, and the Lie algebra structure is obtained by taking the pointwise bracket. This is a subalgebra of the loop algebra

$$L(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}].$$

Let $L_1^\sigma(\mathfrak{g})$ be the kernel of the evaluation map $L^\sigma(\mathfrak{g}) \to \mathfrak{g}$ at $1 : x \otimes a(t) \mapsto a(1)\, x$. Similarly, by $L_1^\sigma(G_o)$, we mean the set of algebraic maps $f : S^1 \to G_o$ with $f(-z) = \sigma(f(z))$ for all $z \in S^1$ and $f(1) = 1$ (where we call a map $f : S^1 \to G_o$ *algebraic* if it extends to an algebraic map $\bar{f} : \mathbb{C}^* \to G$).

We recall the following result from [11, Theorem 1.6].

Theorem 4.3 *Appropriately defined, the integration map defines an algebra isomorphism in cohomology*

$$H^*(L^\sigma(\mathfrak{g}), \mathfrak{k}) \simeq H^*(\mathcal{X}_\sigma).$$

Similarly, we have an algebra isomorphism

$$H^*(L_1^\sigma(\mathfrak{g})) \simeq H^*(L_1^\sigma(G_o)),$$

where $L_1^\sigma(G_o)$ is endowed with the Hausdorff topology induced from an ind-variety structure.

Analogous to the result of Garland and Raghunathan [8], we have the following.

Theorem 4.4 *The inclusion $L_1^\sigma(G_o) \hookrightarrow \Omega_1^\sigma(G_o)$ is a homotopy equivalence, where $L_1^\sigma(G_o)$ is endowed with the Hausdorff topology as in the previous theorem and $\Omega_1^\sigma(G_o)$ is equipped with the compact-open topology.*

Similarly, the projective ind-variety $\mathcal{X}_\sigma$ under the Hausdorff topology is homotopically equivalent with the space $\Omega^\sigma(G_o)/K_o$.

For any invariant homogeneous polynomial $P \in S^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ of degree $d+1$ ($d \geq 1$), define the map

$$\phi_P : \wedge_{\mathbb{C}}^{2d}(L(\mathfrak{g})) \to \mathbb{C}$$

by

$$\phi_P(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d-1}) = \frac{1}{\pi i} \int_{\theta=0}^{\pi} \Phi_P(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d-1}),$$

where $\Phi_P : \wedge_{\mathbb{C}}^{2d}(L(\mathfrak{g})) \to \Omega^1$ is the map defined by

$$\Phi_P\,(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d-1}) := \sum_{\mu \in S_{2d}} \varepsilon(\mu)\, P\left(v_{\mu(0)}, \left[v_{\mu(1)}, v_{\mu(2)}\right], \ldots, \left[v_{\mu(2d-3)}, v_{\mu(2d-2)}\right], dv_{\mu(2d-1)}\right).$$

Here Ω^1 is the space of algebraic 1-forms on $\mathbb{C}^*$, $d(x \otimes a(t)) = x \otimes a'(t)dt$ (for $x \in \mathfrak{g}$ and $a(t) \in \mathbb{C}[t, t^{-1}]$) and in the integral $\int_{\theta=o}^{\pi}$ we make the substitution $t = e^{i\theta}$.

Let $\pi_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ be the projection under the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We similarly define $\pi_{\mathfrak{p}}$. Define the $\mathfrak{k}$-invariant map (for any $P \in S^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$)

$$\hat{\phi}_P : \wedge^{2d}_{\mathbb{C}} \left(L^\sigma(\mathfrak{g})/\mathfrak{k}\right) \to \mathbb{C}$$

by

$$\hat{\phi}_P(\bar{v}_0 \wedge \cdots \wedge \bar{v}_{2d-1}) = \phi_P(v^o_0 \wedge \cdots \wedge v^o_{2d-1}),$$

where $\bar{v}_i := v_i + \mathfrak{k} \in L^\sigma(\mathfrak{g})/\mathfrak{k}$ and $v^o_i := v_i - v_i(1)$. Then, $\hat{\phi}_P$ can be viewed as a cochain for the Lie algebra pair $(L^\sigma(\mathfrak{g}), \mathfrak{k})$.

Lemma 4.5 *Let $P \in S^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$. Then, for the differential δ in the standard cochain complex of the pair $(L^\sigma(\mathfrak{g}), \mathfrak{k})$, $\delta\hat{\phi}_P$ descends to a cocycle for the Lie algebra pair $(\mathfrak{g}, \mathfrak{k})$ under the evaluation map $L^\sigma(\mathfrak{g}) \to \mathfrak{g}$ at* 1.

Proof Observe first that, by [7], the following diagram is commutative up to a nonzero scalar multiple (i.e., $d \circ \beta_P = z^{-1}\Phi_P\partial$, for some $z \in \mathbb{C}^*$).

$$\begin{tikzcd}
\wedge^{2d+1}_{\mathbb{C}}(L(\mathfrak{g})) \arrow[r, "\beta_P"] \arrow[d, "\partial"] & \Omega^0 \arrow[d, "d"] \\
\wedge^{2d}_{\mathbb{C}}(L(\mathfrak{g})) \arrow[r, "\Phi_P"'] & \Omega^1,
\end{tikzcd}$$

where

$$\beta_P(v_0 \wedge \cdots \wedge v_{2d}) := \sum_{\mu \in S_{2d+1}} \varepsilon(\mu)\, P\left(v_{\mu(0)}, \left[v_{\mu(1)}, v_{\mu(2)}\right], \ldots, \left[v_{\mu(2d-1)}, v_{\mu(2d)}\right]\right),$$

Ω^0 is the space of algebraic functions on $\mathbb{C}^*$, d is the standard deRham differential, and ∂ is the standard differential in the chain complex of the Lie algebra $L(\mathfrak{g})$. Thus, for $v_i \in L(\mathfrak{g})$,

$$\begin{aligned}(\delta\phi_P)(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d}) &= \frac{1}{\pi i} \int_{\theta=0}^{\pi} \Phi_P\left(\partial(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d})\right) \\ &= \frac{z}{\pi i} \int_{\theta=0}^{\pi} d\left(\beta_P(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d})\right) \\ &= \frac{z}{\pi i} \left(\beta_P(v_0(-1) \wedge \ldots \wedge v_{2d}(-1))\right. \\ &\qquad \left. -\beta_P(v_0(1) \wedge \cdots \wedge v_{2d}(1))\right). \end{aligned} \tag{1}$$

We next show that for any $v_0, \ldots, v_{2d} \in L^\sigma(\mathfrak{g})$,

(2) $$(\delta\hat{\phi}_P)(\bar{v}_0 \wedge \cdots \wedge \bar{v}_{2d}) = (\delta\phi_P)(v_0^o \wedge \cdots \wedge v_{2d}^o),$$

where $\bar{v}_i$ and v_i^o are defined above the statement of this lemma. For any $x, y \in L(\mathfrak{g})$,

(3) $$[x, y]^o - [x^o, y^o] = [x(1), y^o] + [x^o, y(1)].$$

Thus,

$$\begin{aligned}
&(\delta\hat{\phi}_P)(\bar{v}_0 \wedge \cdots \wedge \bar{v}_{2d}) - \delta\phi_P(v_0^o \wedge \cdots \wedge v_{2d}^o)\\
&= \sum_{i<j}(-1)^{i+j}\,\phi_P\Big(\big([v_i, v_j]^o - [v_i^o, v_j^o]\big) \wedge v_0^o \wedge \cdots \wedge \widehat{v_i^o} \wedge \cdots\\
&\quad \wedge \widehat{v_j^o} \wedge \cdots \wedge v_{2d}^o\Big)\\
&= \sum_{i<j}(-1)^{i+j}\,\phi_P(([v_i(1), v_j^o] + [v_i^o, v_j(1)])\\
&\quad \wedge v_0^o \wedge \cdots \wedge \widehat{v_i^o} \wedge \cdots \wedge \widehat{v_j^o} \wedge \cdots \wedge v_{2d}^o), \text{ by (3)}\\
&= \sum_{i<j}(-1)^{i+j}\,\phi_P\Big([v_i(1), v_j^o] \wedge v_0^o \wedge \cdots \wedge \widehat{v_i^o} \wedge \cdots \wedge \widehat{v_j^o} \wedge \cdots \wedge v_{2d}^o\Big)\\
&\quad + \sum_{i>j}(-1)^{i+j}\,\phi_P\Big([v_j^o, v_i(1)] \wedge v_0^o \wedge \cdots \wedge \widehat{v_j^o} \wedge \cdots \wedge \widehat{v_i^o} \wedge \cdots \wedge v_{2d}^o\Big)\\
&= \sum_i (-1)^i\,(v_i(1)\cdot\phi_P)\Big(v_0^o \wedge \cdots \wedge \widehat{v_i^o} \wedge \cdots \wedge v_{2d}^o\Big)\\
&= 0, \quad \text{since } \phi_P \text{ is } \mathfrak{g}\text{-invariant.}
\end{aligned}$$

This proves (2).

In particular, for any $v_0 \in L^\sigma(\mathfrak{g})$ such that $v_0(1) = 0$ and $v_1, \ldots, v_{2d} \in L^\sigma(\mathfrak{g})$, we get (by using (1)–(2))

$$\begin{aligned}
\delta\hat{\phi}_P\,(\bar{v}_0 \wedge \cdots \wedge \bar{v}_{2d}) &= \delta\phi_P\,\big(v_0 \wedge v_1^o \wedge \cdots \wedge v_{2d}^o\big)\\
&= \frac{z}{\pi i}\beta_P\,\big(v_0(-1) \wedge v_1^o(-1) \wedge \cdots \wedge v_{2d}^o(-1)\big), \text{ since } v_0(1) = 0\\
&= 0, \text{ since } v_0(-1) = \sigma(v_0(1)) = 0.
\end{aligned}$$

This proves the lemma. □

By Identity (1) of the above lemma, the restriction $\bar{\phi}_P$ of ϕ_P to $\wedge_{\mathbb{C}}^{2d}(L_1^\sigma(\mathfrak{g}))$ is a cocycle (for the Lie algebra $L_1^\sigma(\mathfrak{g})$).

As is well known, $S(\mathfrak{g}^*)^{\mathfrak{g}}$ is freely generated by certain homogeneous polynomials $P_1, \ldots, P_{\ell_{\mathfrak{g}}}$ of degrees $m_1 + 1, m_2 + 1, \ldots, m_{\ell_{\mathfrak{g}}} + 1$ respectively, where $\ell_{\mathfrak{g}}$ is the rank of $\mathfrak{g}$ and $m_1 = 1 < m_2 \leq \cdots \leq m_{\ell_{\mathfrak{g}}}$ are the exponents of $\mathfrak{g}$.

The following result is obtained by combining [15, Proposition 4.11.3] and Theorems 4.3 and 4.4.

Theorem 4.6 *The cohomology classes* $[\bar{\phi}_{P_1}], \ldots, [\bar{\phi}_{P_{\ell_{\mathfrak{g}}}}] \in H^*(L_1^\sigma(\mathfrak{g}))$ *freely generate the algebra*

$$H^*\big(L_1^\sigma(\mathfrak{g})\big) \simeq H^*\big(L_1^\sigma(G_o)\big) \simeq H^*\big(\Omega_1^\sigma(G_o)\big).$$

Define the differential graded algebra (for short DGA)

$$\mathcal{D} = H^*(L_1^\sigma(\mathfrak{g})) \otimes C^*(\mathfrak{g}, \mathfrak{k})$$

under the graded tensor product algebra structure. We define the differential d in $\mathcal{D}$ as follows: Take $d_{|C^*(\mathfrak{g},\mathfrak{k})}$ as the standard differential δ of the cochain complex $C^*(\mathfrak{g}, \mathfrak{k})$ of the Lie algebra pair $(\mathfrak{g}, \mathfrak{k})$ and $d([\bar{\phi}_{P_i}]) = \delta\hat{\phi}_{P_i}$ (cf. Lemma 4.5). There is a differential graded algebra homomorphism $\mu : \mathcal{D} \to C^*(L^\sigma(\mathfrak{g}), \mathfrak{k})$ defined by

$$\mu\left([\bar{\phi}_{P_i}]\right) = \hat{\phi}_{P_i}$$

and $\mu_{|C^*(\mathfrak{g},\mathfrak{k})}$ is the canonical inclusion $j : C^*(\mathfrak{g}, \mathfrak{k}) \subset C^*(L^\sigma(\mathfrak{g}), \mathfrak{k})$ under the evaluation map at 1.

Applying the Hirsch lemma to the fibration given in the beginning of this section:

$$\Omega_1^\sigma(G_o) \to \Omega^\sigma(G_o)/K_o \xrightarrow{\gamma} G_o/K_o,$$

(cf. [5, Lemma 3.1]), and using Theorems 4.3, 4.4 and 4.6, we get the following.

Theorem 4.7 *The map μ induces a graded algebra isomorphism in cohomology*

$$[\mu] : H^*(\mathcal{D}) \xrightarrow{\sim} H^*(\mathcal{X}_\sigma).$$

In particular, by Corollary 4.2, any cohomology class $[x] \in H^*(\mathcal{X}_\sigma)$ can be represented by a cocycle $x \in C^*(L^\sigma(\mathfrak{g}), \mathfrak{k})$ of the form

$$x = \sum_{\mathfrak{i}=(i_1,\ldots,i_{\ell_\mathfrak{g}})\in\mathbb{Z}_+^{\ell_\mathfrak{g}}} j\left(c(Q_\mathfrak{i})\right)(\hat{\phi}_{P_1})^{i_1}\cdots(\hat{\phi}_{P_{\ell_\mathfrak{g}}})^{i_{\ell_\mathfrak{g}}},$$

for some $Q_\mathfrak{i} \in S(\mathfrak{k}^*)^\mathfrak{k}$, where $c : S(\mathfrak{k}^*)^\mathfrak{k} \to C(\mathfrak{g}, \mathfrak{k})$ is the Chern–Weil homomorphism defined in the beginning of this section.

Finally, we are ready to prove the second main theorem of this paper.

Theorem 4.8 *Let $\mathfrak{g}$ be a simple Lie algebra and let σ be an involution of $\mathfrak{g}$ with $+1$ (resp. -1) eigenspace $\mathfrak{k}$ (resp. $\mathfrak{p}$). Assume that $\mathfrak{p}$ is an irreducible $\mathfrak{k}$-module. Then, the algebra $A^\mathfrak{k}$ of $\mathfrak{k}$-invariants of A is generated (as an algebra) by the element S, where A and S are defined in the Introduction.*

In particular, $(A^\mathfrak{k})^{p,q} = 0$ if $p \neq q$.

Proof By Theorem 3.1, the algebra $B^\mathfrak{k}$ is graded isomorphic with the singular cohomology $H^*(\mathcal{Y})$, where $B := \frac{\wedge(\mathfrak{p})\otimes\wedge(\mathfrak{p})}{\langle C_1 \oplus C_2\rangle}$. Moreover, the inclusion $a : \mathcal{Y} \subset \mathcal{X}_\sigma$ induces a surjection in cohomology, since $\mathcal{X}_\sigma$ is obtained from $\mathcal{Y}$ by attaching real even-dimensional cells (by virtue of the Bruhat decomposition). Thus, we have

$$H^*(\mathcal{X}_\sigma) \overset{a^*}{\twoheadrightarrow} H^*(\mathcal{Y}) \xrightarrow[\sim]{\xi} B^\mathfrak{k} \xrightarrow{\eta} A^\mathfrak{k},$$

where $\eta : B^\mathfrak{k} \to A^\mathfrak{k}$ is the standard quotient map.

By Theorem 4.7, any cohomology class $[x] \in H^*(\mathcal{X}_\sigma)$ can be represented by a cocycle $x \in C^*(L^\sigma(\mathfrak{g}), \mathfrak{k})$ of the form

$$x = \sum_{\mathfrak{i}=(i_1,\ldots,i_{\ell_{\mathfrak{g}}})\in\mathbb{Z}_+^{\ell_{\mathfrak{g}}}} j\,(c(Q_{\mathfrak{i}}))\,(\hat{\phi}_{P_1})^{i_1}\cdots(\hat{\phi}_{P_{\ell_{\mathfrak{g}}}})^{i_{\ell_{\mathfrak{g}}}},$$

for some $Q_{\mathfrak{i}} \in S(\mathfrak{k}^*)^{\mathfrak{k}}$.

If $Q_{\mathfrak{i}}$ has constant term 0, from the definition of the Chern–Weil homomorphism c, it is clear that under the composite map $\mathfrak{y} := \eta \circ \xi \circ a^*$, $j(c(Q_{\mathfrak{i}}))$ goes to zero. Further, by an argument similar to the proof of Theorem 2.8 in [13], we see that $\hat{\phi}_{P_i}$ goes to zero under $\mathfrak{y}$ for any $2 \le i \le \ell_{\mathfrak{g}}$. We briefly recall the main argument here.

For any $\mu \in S_{2d}$ and $P \in S^{d+1}(\mathfrak{g}^*)^{\mathfrak{g}}$ $(d \ge 2)$, consider the linear form

$$\hat{\phi}_{P,\mu} : \otimes_{\mathbb{C}}^{2d}\left(L^\sigma(\mathfrak{g})/\mathfrak{k}\right) \to \mathbb{C},$$

defined by

$$\begin{aligned}&\hat{\phi}_{P,\mu}\,(\bar{v}_0 \otimes \bar{v}_1 \otimes \cdots \otimes \bar{v}_{2d-1})\\ &\quad= \int_{\theta=0}^{\pi} P\left(v^o_{\mu(0)}, \left[v^o_{\mu(1)}, v^o_{\mu(2)}\right], \ldots, \left[v^o_{\mu(2d-3)}, v^o_{\mu(2d-2)}\right], dv^o_{\mu(2d-1)}\right),\end{aligned}$$

where $\bar{v}_i := v_i + \mathfrak{k}$. For the notational convenience, assume $\mu(1) < \mu(2)$. For any fixed

$$v_0, v_1, \ldots, \hat{v}_{\mu(1)}, \ldots, \hat{v}_{\mu(2)}, \ldots, v_{2d-1} \in L^\sigma(\mathfrak{g}),$$

consider the restriction $\bar{\phi}_{P,\mu}$ of the function $\hat{\phi}_{P,\mu}$ to

$$\bar{v}_0 \times \bar{v}_1 \times \cdots \times \oplus_{p\in\mathbb{Z}}\left(\mathfrak{g}_{2p+1} \otimes t^{2p+1}\right) \times \cdots \times \oplus_{p\in\mathbb{Z}}\left(\mathfrak{g}_{2p+1} \otimes t^{2p+1}\right) \times \cdots \times \bar{v}_{2d-1},$$

where the two copies of $\oplus_{p\in\mathbb{Z}}\left(\mathfrak{g}_{2p+1} \otimes t^{2p+1}\right)$ are placed in the $\mu(1)$ and $\mu(2)$th slots. Then, under the identification $\mathfrak{g}_p \otimes t^p \cong (\mathfrak{g}_p \otimes t^p)^*$ induced from the bilinear form $\langle\ ,\ \rangle$,

$$\begin{aligned}\bar{\phi}_{P,\mu} &= \sum_{i,j,m,n} f_i(n) \otimes f_j(m) \int_{\theta=0}^{\pi} P\left(v^o_{\mu(0)}, \left[e_i(n)^o, e_j(m)^o\right], \left[v^o_{\mu(3)}, v^o_{\mu(4)}\right], \ldots,\right.\\ &\qquad\left.\left[v^o_{\mu(2d-3)}, v^o_{\mu(2d-2)}\right], dv^o_{\mu(2d-1)}\right)\\ &= \sum_{i,j,m,n,k'} f_i(n) \otimes f_j(m) \int_{\theta=0}^{\pi} P\left(-, \langle[e_i, e_j], e'_{k'}\rangle F_{k'}(n,m), -\right)\end{aligned}$$

$$= \sum_{i,j,m,n,k'} \langle e_i, [e_j, e'_{k'}] \rangle \, f_i(n) \otimes f_j(m) \int_{\theta=0}^{\pi} P\left(-, F_{k'}(n,m), -\right)$$

$$= \sum_{j,k',m,n} [e_j, e'_{k'}](n) \otimes f_j(m) \int_{\theta=0}^{\pi} P\left(-, F_{k'}(n,m), -\right)$$

$$= -\sum_{j,k',m,n} [e'_{k'}, e_j](n) \otimes f_j(m) \int_{\theta=0}^{\pi} P\left(-, F_{k'}(n,m), -\right),$$

where, as in the Introduction, $\{e_i\}$ is a basis of $\mathfrak{p}$ and $\{f_i\}$ is the dual basis; $\{e'_{k'}\}$ is a basis of $\mathfrak{k}$ and $\{f'_{k'}\}$ is the dual basis; m, n run over the odd integers and $F_{k'}(n,m) := f'_{k'}(n+m) - f'_{k'}(n) - f'_{k'}(m) + f'_{k'}$.

Thus, only the powers of $\hat{\phi}_{P_1}$ contribute to the image of $\mathfrak{y}$. This completes the proof of the theorem. □

Remark 4.9 It is likely that for the validity of Theorem 4.8 it is enough to assume that $\mathfrak{g}$ is semisimple (not necessarily simple). However, we must assume that $\mathfrak{p}$ is $\mathfrak{k}$-irreducible under the adjoint action since the second grade component $(A^2)^{\mathfrak{k}}$ has dimension at least equal to the number of irreducible components of the $\mathfrak{k}$-module $\mathfrak{p}$.

Acknowledgments It is my pleasure to thank Weiqiang Wang for asking the question answered in this paper. I also thank Pierluigi Frajria and Paolo Papi for some helpful correspondences. In particular, the proof of Lemma 2.4 is due to them. This work was partially supported by the FRG grant no DMS-0554247 from NSF.

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