Quantization and chiralization

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1 Four fundamental theories:

$$\begin{tikzcd}
\mathrm{ClFT} \arrow[d] & & \mathrm{QFT} \arrow[ll] \arrow[d] \\
\mathrm{ClM} \arrow[rr, dashed, bend right, "\text{quantization}"'] & & \mathrm{QM} \arrow[ll] \arrow[u, dashed, bend right, "\text{chiralization}"']
\end{tikzcd}$$

Corresponding algebraic structures:

$$\begin{tikzcd}
\text{Poisson vertex algebras} \arrow[d] & \text{Vertex algebras} \arrow[l] \arrow[d] \\
\text{Poisson algebras} & \text{Associative algebras} \arrow[l]
\end{tikzcd}$$

(2) Quasiclassical limit for associative algebras

Given a family of unital associative algebras (over $\mathbb{C}$) $A_\hbar$, i.e. an unital associative algebra over $\mathbb{C}[[\hbar]]$ s.t. multiplication by $\hbar$ is injective, and

$$[A_\hbar, A_\hbar] \subset \hbar A_\hbar ,$$

the **quasiclassical limit** is

$$\mathcal{A} = A_\hbar / \hbar A_\hbar$$

with the induced associative product, and the **Poisson bracket**

$$\{a, b\} = \{\tilde{a}, \tilde{b}\}^\sim \bmod \hbar A_\hbar , \text{ where}$$

$$[\tilde{a}, \tilde{b}] \equiv \tilde{a}\tilde{b} - \tilde{b}\tilde{a} = \hbar\{\tilde{a}, \tilde{b}\}^\sim \in \hbar A_\hbar \subset A_\hbar ,$$

$\tilde{a}, \tilde{b}$ are some preimages of $a, b \in \mathcal{A}$ in A.

We thus get a **Poisson algebra** $\mathcal{A}$, i.e. a unital commutative associative algebra with $\{\ ,\ \}$, satisfying Lie algebra axioms + the Leibniz rule:

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

③ Wightman's axioms of QFT (rough algebraic version).

1950s

Data:

- V space of states (vector space $/\mathbb{C}$)
- $|0\rangle \in V$ vacuum vector
- a representation π of the Poincaré group (= group of isometries of the Minkowski space M^D) in the space V
- a collection of quantum fields (= End V-valued distributions $\{\Phi^\alpha(z)\}_\alpha$ on M^D such that $\phi^\alpha(z)v$ is a V-valued function, $v \in V$)

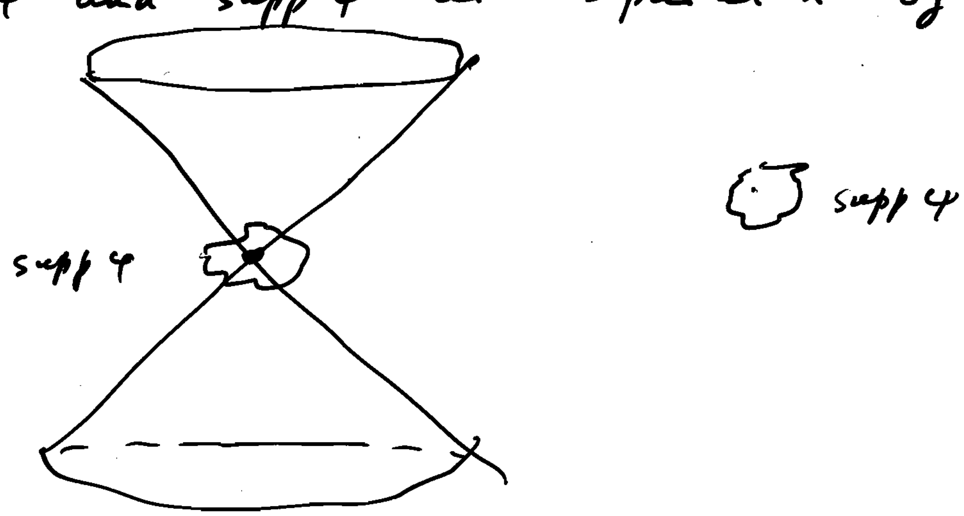
Axioms†: (vacuum) $|0\rangle$ is fixed by the Poincaré group

(Poincaré covariance) $\pi(g)\,\phi^\alpha(z)\,\pi(g)^{-1} = \phi^{g\cdot\alpha}(g\cdot z)$

(completeness) $\phi^{\alpha_1}(\varphi_1)\dots\phi^{\alpha_s}(\varphi_s)|0\rangle$ span V

(locality) $[\phi^\alpha(\varphi), \phi^\beta(\psi)] = 0$ if

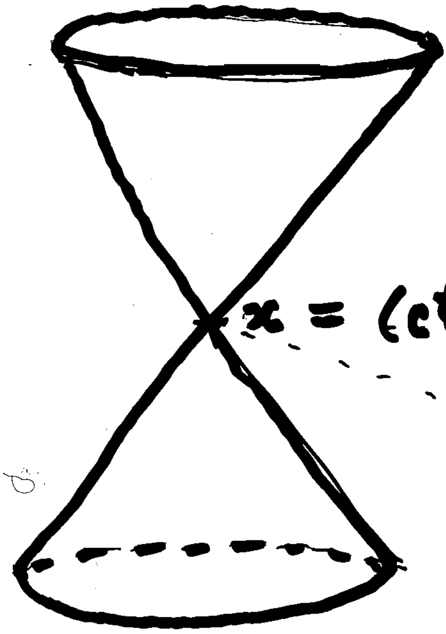
supp φ and supp ψ are separated by light cone:



Minkowski space-time $M^D = \mathbb{R}^D$

metric $|x|^2 = x_0^2 - x_1^2 - \ldots - x_{D-1}^2$

$x_0 = ct$; $x_1, \ldots, x_{D-1}$ space coordinates



light cone $|x|^2 = 0$

$x = (ct, x_1, \ldots, x_{D-1})$

$y = (ct', y_1, \ldots, y_{D-1})$

$|x-y| < 0$, i.e. $\dfrac{\sqrt{\sum_{i=1}^{D-1}(x_i - y_i)^2}}{t - t'} > c$:

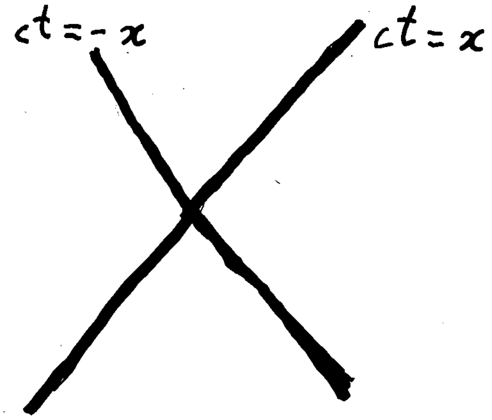
the speed of a signal travelling from x to y should be $> c \Rightarrow$

$$\phi^\alpha(x)\,\phi^\beta(y) = \phi^\beta(y)\,\phi^\alpha(x)\,,$$ i.e.

measurements at x and y are independent

(relativity + uncertainty principle)

④ If $D = 2$, the light cone is:



2 lines

Chiral part of 2D QFT =

take quantum fields supported on one of the lines

Then Poincaré group = group of translations of a 1-dim space + dilations

↖ ignore (but later bring back)

and representation of its Lie algebra is a single operator $T \in \operatorname{End} V$

Thus we arrive at the definition of a vertex algebra (= chiral algebra)

"Vertex algebras for beginners" AMS 1996, 1998

<u>Remark</u> <u>One can't cancel</u> $(z-w)^N$, e.g.:

<u>Ex2</u>. $\delta(z,w) = z^{-1}\sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n$; $(z-w)\delta(z,w) = 0$

But this is basically all that can happen, namely, locality $\Leftrightarrow$

<u>Ex3</u> OPE $\quad [\phi^\alpha(z), \phi^\beta(w)] = \sum_{j=0}^{N-1} \varphi^j(w) \left(\frac{\partial}{\partial w}\right)^j \delta(z,w)/j!$

(interaction of fields ϕ^α and ϕ^β produces new fields φ^j)

In order to keep track of this, introduce λ-bracket:

$$[\phi^\alpha{}_\lambda\, \phi^\beta] = \sum_{j=0}^{N-1} \frac{\lambda^j}{j!}\varphi^j,$$

It encodes the "singular part" of OPE and satisfies axioms, similar to that of a Lie algebra. This new structure is called a Lie conformal algebra

These correspond ~bijectively to formal distribution Lie algebras.

Vertex algebra:

Data: $\cdot$ V space of states = vector space / $\mathbb{C}$

$\cdot$ $|0\rangle \in V$ vacuum vector

$\cdot$ $T \in \mathrm{End}\, V$ translation operator

$\cdot$ $\mathcal{F} = \{ \Phi^\alpha(z) = \sum_{n \in \mathbb{Z}} \Phi^\alpha_{(n)} z^{-n-1} \}_\alpha$

a collection of $\mathrm{End}\, V$-valued quantum fields (i.e. $\Phi^\alpha_{(n)} \in \mathrm{End}\, V$, $\Phi^\alpha_{(n)} v = 0$ for $n \gg 0$).

Axioms: (vacuum) $T|0\rangle = 0$,

(translation covariance) $[T, \Phi^\alpha(z)] = \frac{d}{dz}\Phi^\alpha(z)$,

(completeness) $\Phi^{\alpha_1}_{(n_1)} \cdots \Phi^{\alpha_s}_{(n_s)} |0\rangle$ span V,

(locality) $(z-w)^N [\Phi^\alpha(z), \Phi^\beta(w)] = 0$ for some $N \in \mathbb{Z}_+$.

"Trivial" example. V = unital commut. assoc. alg.
$|0\rangle = 1$, $T = 0$, $\mathcal{F} = \{L_a\}_{a \in V}$

Exercise 1. VA with $T = 0$ are precisely unital commutative associative algebras

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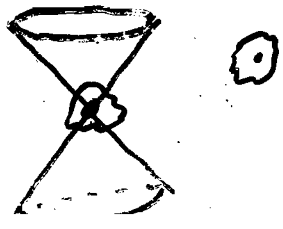
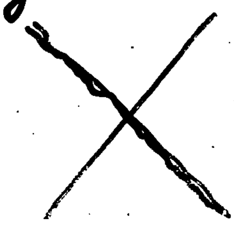
Comment: relation to Wightman axioms:
(VA for beginners)



$D = 2$



chiral part

Example 1. (affine) $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] + \mathbb{C}K$ (simple Lie alg. — $\mathfrak{g}$; central — K)

$$(*[at^m, bt^n] = [a,b]t^{m+n} + m\delta_{m,-n}(a,b)K)$$

is a formal distribution Lie algebra,

with $\mathcal{F} = \{a(z) = \sum_n (at^n) z^{-n-1}\}_{a\in\mathfrak{g}}$

$$* \ [a(z), b(w)] = [a,b](w)\,\delta(z-w) + (a,b)K\,\delta'_w(z-w)$$

or $* \quad [a_\lambda b] = [a,b] + \lambda(a,b)K$ is the

λ-bracket of a Lie conf algebra $\mathbb{C}[T]\mathfrak{g} + \mathbb{C}K$

Example 2. (Virasoro) $! \, [L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C$

$$\mathcal{F} = \{L(z) = \sum_n L_n z^{-n-2}\}$$

$$! \quad [L_\lambda L] = (T+2\lambda)L + \frac{\lambda^3}{12}C$$

λ-bracket on the Vir Lie conf alg. $\mathbb{C}[T]L + \mathbb{C}C$

Example 3. (free fermions) A vector superspace with a non-degenerate skewsymmetric bilinear form $\langle .,. \rangle$

$$\hat{A} = A[t,t^{-1}] + \mathbb{C}\mathbb{1}$$

$$[at^m, bt^n] = \langle a,b \rangle \delta_{m,-n-1}\mathbb{1}$$

$$\mathcal{F} = \{a(z) = \sum_n (at^n) z^{-n-1}\}_{a\in A}$$

$$[a_\lambda b] = \langle a,b \rangle \mathbb{1} \quad \text{on } \mathbb{C}[T]A + \mathbb{C}\mathbb{1}$$

Correspondence: $R \longleftrightarrow \mathrm{Lie}\,R \supset \underbrace{(\mathrm{Lie}\,R)_-}_{\text{annihilation subalg}}$ (non-negative powers of t)

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Ex 4. Lemma. $\Phi^\alpha(z)|0\rangle \in V[[z]]$

(In general, by def., $\Phi^\alpha(z)v \in V((z))$.)

Hence have field-state correspondence:

$$\Phi^\alpha(z) \longmapsto \Phi^\alpha := \Phi^\alpha(z)|0\rangle\Big|_{z=0} \in V$$

Extension Theorem. Let $\overline{\mathcal{F}}$ be the collection of all translation covariant quantum fields, which are local to all fields from $\mathcal{F}$. Then

(a) all axioms of VA still hold;

(b) the field-state correspondence

$$\overline{\mathcal{F}} \to V, \quad \Phi(z) \mapsto \Phi = \Phi(z)|0\rangle\Big|_{z=0}$$

is bijective.

This leads, to the second, equivalent definition of a VA

2nd definition

<u>Vertex algebra</u> is $(V, |0\rangle, T, \underset{V}{\overset{}{\phi}} \xmapsto{\text{state-field corr.}} \phi(z))$

subject to the following axioms:

(vacuum) $T|0\rangle = 0$, $\phi(z)|0\rangle\big|_{z=0} = \phi$

(translation covariance) $[T, \phi(z)] = \frac{d}{dz}\phi(z)$

(locality) $(z-w)^N[\phi(z), \psi(w)] = 0$, some $N \in \mathbb{Z}_+$

<u>Remark</u>. Completeness automatic: $\phi_{(-1)}|0\rangle = \phi$.

<u>Remark</u>. Each quantum field $\phi(z) = \sum_{n\in\mathbb{Z}} \phi_{(n)} z^{-n-1}$, where $\phi_{(n)} \in \operatorname{End} V$. Hence for each $n \in \mathbb{Z}$ can define n-th product on V:

$$\phi_{(n)}\psi = \phi_{(n)}(\psi) \quad (\phi, \psi \in V) \quad n \in \mathbb{Z}$$

3d

<u>Original Borchards' definition</u> (1986): (use uniqueness thm)

V is an algebra with infinitely many products $\underset{n\in\mathbb{Z}}{} a_{(n)}b \; (= 0 \text{ for } n \gg 0)$ and $|0\rangle \in V$ subject to axioms:

(vacuum) $|0\rangle_{(n)} a = \delta_{n,-1} a \; (n \in \mathbb{Z})$; $a_{(n)}|0\rangle = \delta_{n,-1} a \; (n \geq -1)$

(Borcherds identity) $\sum_{j\in\mathbb{Z}_+} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)} c$

$$= \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{n}{j} \left(a_{(m+n-j)}(b_{(k+j)}c) - (-1)^n b_{(n+k-j)}(a_{(m+j)}c)\right)$$

$a, b, c \in V$; $m, n, k \in \mathbb{Z}$.

⑥ Denote

$:ab: = a_{(-1)}b$ (normally ordered product)

$[a_\lambda b] = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!} a_{(n)} b$ (λ-bracket) polynomial in λ

4th equivalent definition of a VA

(Poisson algebra like definition) (Bakalov-VK math.QA /0204282)

Data: $(V, |0\rangle, T, :ab:, [a_\lambda b])$

Axioms:

Ⓐ $(V, |0\rangle, T, :ab:)$ is a unital, differential algebra, (i.e. T is a derivation of all products) "quasicommutative" and "quasiassociative":

(quasicommutativity) $:ab: - :ba: = \int_{-T}^{0} [a_\lambda b] d\lambda$

(quasiassociativity) $::ab:c: - :a:bc::$ = expression via λ-bracket, symm. w.r.to $a \leftrightarrow b$

Ⓑ $(V, T, [a_\lambda b])$ is a Lie conformal algebra

(sesquilinearity) $[Ta_\lambda b] = -\lambda [a_\lambda b]$, T derivation

(skewsymmetry) $[a_\lambda b] = -[b_{-\lambda-T} a]$,

(Jacobi) $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]]$

Ⓒ Quasi Leibniz rule:

$[a_\lambda :bc:] = :[a_\lambda b]c: + :b[a_\lambda c]: + \underbrace{\int_0^\lambda [[a_\lambda b]_\mu c] d\mu}_{\text{"quantum corr."}}$

<u>Remark.</u> One can combine quasicommutativity and quasiassociativity in one axiom:

$$:a:bc:: - :b:ac:: = \underbrace{:\left(\int_{-T}^{0} [a_\lambda b]\, d\lambda\right) c:}_{\text{"quantum correction"}}$$

(left multiplication is quasicommutative).

Poisson vertex algebra

Its definition is obtained from the 4th definition of a VA by deleting "quantum corrections" in (A) and (C):

Data: $(V, |0\rangle, T, ab, \{a_\lambda b\})$

Axioms:

(A) $(V, |0\rangle, T, ab)$ is a unital differential commutative associative algebra

(B) $(V, T, \{a_\lambda b\})$ is a Lie conformal algebra

(C) Leibniz rule:

$$\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\} \quad .$$

Main examples = quasiclassical limit of a family of vertex algebras

(8) The theory of infinite-diml integrable systems

<u>Remark.</u> If V is a Lie conformal algebra, then $\mathrm{Lie}\,V := V/TV$ with bracket

$$[a, b] = [a_\lambda b]|_{\lambda=0}$$

is a Lie algebra, and V is a Lie V-mod:

$$a \cdot v = [a_\lambda v]|_{\lambda=0}.$$

Local functionals: are elements of $\mathrm{Lie}\,V$.

Hamiltonian equation corr. to local functional $\underline{h}$ $\}$: $\dot{u} = [h, u]$, $u \in V$.

It is completely integrable if there are infinitely many (lin. independent) integrals of motion $h_i \in \mathrm{Lie}\,V$ in involution

i.e. $[h, h_i] = 0$ ($\Leftrightarrow \dot{h}_i = 0$) s.t. $[h_i, h_j] = 0$

If V is a PVA, get a <u>classical</u> Hamiltonian equation, if V is a VA, get a <u>quantum</u> Hamiltonian equation.

Example : KdV

Poisson VA $\mathcal{V} = \mathbb{C}[u, u', u'', \dots]$,

$|0\rangle = 1$, $T u^{(n)} = u^{(n+1)}$ derivation

$\{u_\lambda u\} = \lambda$ Gardner - Faddeev - Zakharov bracket GFZ (1971)
extended by Leibniz rule, sesquilinearity and skewsym.

Consider the local functionals:

$h_0 = \int u$, $h_1 = \int u^2$, $h_2 = \frac{1}{2}\int (u^3 - u'^2)$, ...

Sign of $\int$ means image in $\mathcal{V}/T\mathcal{V}$.

Ex 5. Check that they are in involution

and that the Hamiltonian eqn

$\dot{u} = \{h_2, u\}$

is KdV : $\underline{\dot{u} = 3uu' + u'''}$.

It is completely integrable.

General Lenard scheme

Barakat - De Sole - Kac (2009)

A challenge to Lie algebraists:

Classify all infinite-dimensional maximal abelian subalgebras of Lie $\mathcal{V}$.

(Unknown even for GFZ)

$\Longleftrightarrow$ classification of all completely integrable Hamiltonian systems of PDE

New example in case of two functions u and v:

$$\begin{cases} \dot{u} = \left(\frac{1}{v}\right)' + c\left(\frac{1}{v}\right)''' \\ \dot{v} = -\left(\frac{u}{v^2}\right)' \end{cases}$$

(9) The simplest non-trivial example of a vertex algebra is the quantization of the GFZ Poisson vertex algebra.

Definition. Let $V_\hbar$ be a family of VA, such that $[V_\hbar\,{}_\lambda\, V_\hbar] \subset \hbar V_\hbar[\lambda]$. Then on $\mathcal{V} = V_\hbar / \hbar V_\hbar$ get the canonical induced structure of a PVA, called the quasiclassical limit of $V_\hbar$.

Example. Free boson VA = quantization of GFZ

$B_\hbar = \mathbb{C}[u, u', u'', \dots][[\hbar]]$

$|0\rangle = 1$, $Tu^{(n)} = u^{(n+1)}$, $\mathcal{F} = \{\alpha(z) = \sum_n \alpha_n z^{-n-1}\}$

where $\alpha_{-n} = n!\, u^{(n-1)}$, $\alpha_n = \frac{\hbar}{(n-1)!} \frac{\partial}{\partial u^{(n-1)}}$,
$(n \geq 1)$

$\alpha_0 = 0$, so that

$[\alpha_m, \alpha_n] = \hbar m\, \delta_{m,-n}$, or $[\alpha(z), \alpha(w)] = \hbar\, \delta'_w(z-w)$

or $[\alpha_\lambda \alpha] = \hbar \lambda$

In the quasiclassical limit:

$$\{\alpha_\lambda \alpha\} = \lim_{\hbar \to 0} \frac{[\alpha_\lambda \alpha]}{\hbar} = \lambda \qquad \text{GFZ bracket}$$

More generally:

Universal enveloping VA $V(R)$

of a Lie conf. algebra R

Def 1. $V(R) = \mathrm{Ind}_{(\mathrm{Lie}\,R)_-}^{\mathrm{Lie}\,R}\, \mathbb{C}$

$\equiv \mathcal{U}(\mathrm{Lie}\,R)/\mathcal{U}(\mathrm{Lie}\,R)\,\mathrm{Lie}\,R_-$

$|0\rangle = 1$, T descends, $\mathcal{F} = R$

use def 1 of VA

Def 2. $V(R) = \mathrm{span}\ {:}a_{i_1} \dots a_{i_s}{:}$,

where $\{a_i\}$ basis of R $\quad i_1 \leq \dots \leq i_s$

use def 4 of VA

Example: $R = \mathbb{C}[T]\alpha + \mathbb{C}$

$[\alpha_\lambda \alpha] = \lambda$

$\mathrm{Lie}\,R = \{[\alpha_m, \alpha_n] = m\,\delta_{m,-n}\}_{m,n \in \mathbb{Z}}$

$(\mathrm{Lie}\,R)_- = \{\alpha_m \mid m \geq 0\}$

$V(R)$ free boson

Family: $[a_\lambda b]_\hbar = \hbar [a_\lambda b]$

Quasiclassical limit: $\mathcal{V} = S_{\mathbb{C}}(R)$

PVA

The most important special cases:

Example 1. $\mathfrak{g}$ simple Lie algebra, $k \in \mathbb{C}$.

$$V^k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[t]+\mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}_k$$

$$= \mathcal{U}(\hat{\mathfrak{g}}) / \mathcal{U}(\hat{\mathfrak{g}})(\mathfrak{g}[t] + \mathbb{C}(K-k))$$

Affine vertex algebra $\quad \mathcal{F} = \{a(z) = \sum_n (at^n) z^{-n-1}\}$

$|0\rangle = 1$, $T = -\frac{d}{dt}$

Example 2. A superspace with a skewsymmetric bilinear form $\langle \cdot, \cdot \rangle$

$$F(A) = \mathrm{Ind}_{A[t]+\mathbb{C}\mathbb{1}}^{\hat{A}} \mathbb{C}_1$$

Free fermions based on A

To construct a family, just replace in $\hat{\mathfrak{g}}$ (resp $\hat{A}$) the bracket by

$$[a, b]_\hbar = \hbar [a, b]$$

(10) How to construct the vertical arrows?

Need, in addition, an energy operator H on V, satisfying:

$$[H, a(z)] = \left(z\frac{d}{dz} + \Delta_a\right) a(z)$$

if $Ha = \Delta_a a$

(This amounts to enlarging the translation group to the whole Poincare group (affine transformations) in the covariance axiom)

Example. If one of the fields of V is the Virasoro field

$$L(z) = \sum_n L_n z^{-n-2},$$

s.t. $L_{-1} = T$, then $\underline{H = L_0}$.

In Example 1: $L = \frac{1}{2(k+h^\vee)} \sum_i :a_i b_i:$ $\quad (a_i, b_j) = \delta_{ij}$

Sugawara construction

In Example 2: $L = \frac{1}{2} \sum_i :(Ta_i) b_i:$

Or simply: $L_0 = -t\frac{d}{dt}$ $\quad \left(L_{-1} = -\frac{d}{dt}\right)$

Introduce the ε-deformed quantum fields $a_\varepsilon(z) = (1+\varepsilon z)^{\Delta_a} a(z)$

if $Ha = \Delta_a a$.

$$= \sum_n a_{(n,\varepsilon)} z^{-n-1}$$

so that we have the ε-deformed nth products:

$$a_{(n,\varepsilon)} b = a_{(n,\varepsilon)}(b) = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_a}{j} \varepsilon^j a_{(n+j)} b$$

Remark. The ε-deformed fields $a_\varepsilon(z)$ form a "gauged VA", satisfying the same axioms as VA, except that the translation covariance axiom changes:

$$[T, a_\varepsilon(z)] = \left(\frac{d}{dz} - \frac{\Delta_a \varepsilon}{1+\varepsilon z}\right) a_\varepsilon(z).$$

Introduce the deformed normally ordered product on V:

$$a * b = a_{(-1,\, \epsilon=1)} b$$

Theorem. (Zhu, 1996) (DLM, 1998) (De Sole, VK, 2006) (a) $J = \operatorname{span}\{((T+H)a) * b \mid a, b \in V\}$ is a 2-sided ideal w.r.to the product $*$.

(b) $Z(V) := (V, *)/J$ is a unital associative algebra, which "controls" (twisted) representation theory of V.

The right vertical arrow is just the canonical projection

$$\begin{array}{c} V \\ \downarrow \\ Z(V) \end{array}$$

How the Zhu algebra $Z(V)$ "controls" Rep V?

<u>Def.</u> A positive energy representation of V is a linear map $V \to \mathrm{End}\, M$ - valued fields : $a \mapsto a^M(z) = \sum_n a_n^M z^{-n-1}$

where $a_n^M \in \mathrm{End}\, M$, satisfying

$|0\rangle \to I_M$ and Borcherds identity.

Also $M = \bigoplus_{j \geq 0} M_j$ such that

$$a_n^M M_j \subset M_{j-n} \quad \text{for all } n, j.$$

(Note: $[H, a_n] = -n\, a_n$ if $a(z) = \sum a_n z^{-k-1}$)

Theorem. The map $V \ni a \mapsto a_0^M|_{M_0}$ induces a representation of $Z(V)$ in M_0.

This gives a functor:

$$\mathrm{PERep}\, V \longrightarrow \mathrm{Rep}\, Z(V),$$

bijective on irreducibles.

For the left vertical arrow, define on a PVA $\mathcal{V}$, the bracket

$$\{a, b\} = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_a - 1}{j} a_{(j)} b \, .$$

<u>Theorem.</u> (De Sole – VK, 2006) (a) $J = \mathrm{span}\{((T+H)a)b\}$ is an ideal for both product and $\{\, , \,\}$.

(b) $\mathcal{Z}(\mathcal{V}) = \mathcal{V}/J$ is a Poisson algebra.

(c) Defining the left arrow $\mathcal{V} \downarrow \mathcal{Z}(\mathcal{V})$ as the canonical map, we obtain the commutative diagramm:

$$\begin{tikzcd}
\mathcal{V} \arrow[d] & V \arrow[l] \arrow[d] \\
\mathcal{Z}(\mathcal{V}) & Z(V) \arrow[l]
\end{tikzcd}$$

Proof. B I says: $v \in M$

$$\sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} \left(a^M_{m+n-j} \, b^M_{k+j-n} - (-1)^n b^M_{k-j} \, a^M_{m+j} \right) v$$

$$= \sum_{j \in \mathbb{Z}_+} \binom{m+\Delta_a-1}{j} \left(a_{(n+j)} b \right)^M_{m+k} v$$

Take $v \in M_0$, so that $a_n v = 0$ for $n > 0$.

Let $m = 1$, $n = -1$, $k = -1$ to get:

$$a^M_0 \, b^M_0 \, v = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_a}{j} \left(a_{(j-1)} b \right)^M_0 v$$

We thus get homom $(V, *) \to \mathrm{End}\, M_0$.

Similarly one shows that $J \to 0$.

This gives the functor.

Proof of the rest is similar, but harder. $\square$

Zhu 1996, De Sole - VK 2005.

Spec $H \subset \mathbb{Z}$ general twisted case

untwisted case

Dong - Li - Mason 1998

Examples of squares

Example 1.

$$\begin{tikzcd}
\text{affine PVA} & \text{affine VA} & \text{Virasoro} \\
\mathcal{V}^k(\mathfrak{g}) \arrow[d] & V^k(\mathfrak{g}) \arrow[l] \arrow[d] & L \arrow[d] \\
S(\mathfrak{g}) & U(\mathfrak{g}) \arrow[l] & C \\
\text{Kirillov-Kostant PA} & & \text{Casimir}
\end{tikzcd}$$

2. Super affine VA

$$\hat{\mathfrak{g}}^{super} = \mathfrak{g}[t, t^{-1}, \theta] + \mathbb{C}K \qquad K = k$$

$$[at^m, bt^n] = [a,b]t^{m+n} + m(a,b)(k+h^\vee)\delta_{m,-n}$$

$$[\bar{a}t^m, \bar{b}t^n] = (k+h^\vee)\delta_{m,-n-1}(a,b)$$

$$[at^m, \bar{b}t^n] = \overline{[a,b]}\, t^{m+n},$$

where $\bar{a} = a\theta$

$$V^k_{super}(\mathfrak{g}) = U(\hat{\mathfrak{g}}^{super})/U(\hat{\mathfrak{g}}^{super})(\mathfrak{g}[t,\theta] + \mathbb{C}(K-k))$$

$$|0\rangle = 1, \quad T = -\frac{d}{dt}, \qquad \mathcal{F} = \{a(z), \bar{a}(z)\}_{a\in\mathfrak{g}}$$

$$L = \frac{1}{2(k+h^\vee)}\Big(\sum_i (:a^i a^i: + :T\bar{a}^i a^i) + \frac{1}{k+h^\vee}\sum_{i,j} :\bar{a}^i [a^i, a^j]\bar{a}^j\Big)$$

$$G = \frac{1}{k+h^\vee}\Big(\sum_i :a^i \bar{a}^i: + \frac{1}{3(k+h^\vee)}\sum_{i,j} :\overline{[a^i, a^j]}\,\bar{a}^i \bar{a}^j:\Big)$$

$\{a^i\}$ orthonormal basis of $\mathfrak{g}$

VK - I. Todorov 1985

They form the Neveu-Schwarz (Lie conformal) algebra:

$$[L_\lambda L] = (T + 2\lambda)L + \frac{c}{12}\lambda^3,$$

$$[L_\lambda G] = (T + \tfrac{3}{2}\lambda)G, \quad [G_\lambda G] = 2L + \frac{\lambda^2}{3}c$$

$$\Big(c = \frac{k \dim \mathfrak{g}}{k+h^\vee} + \frac{\dim \mathfrak{g}}{2}\Big)$$

Remark. Guiding principle in finding formulas CONFORMAL WEIGHT

Example 2. (De Sole-VK. 05)

Classical KT

$\hat{\mathfrak{g}}^{super} = \mathfrak{g}[t, t^{-1}, \theta] + \mathbb{C}K$

KT construction of Neveu-Schwarz algebra (VK-Todorov, CMP, 1985)

$NS = \{L, G\}$

Papi-Moseneder-VK 07-09

Weil algebra $W_{\mathfrak{g}}$

→ H. Cartan model (1950) for equivariant differential forms on a manifold:

$H_G(M) = H((W_{\mathfrak{g}} \otimes \Omega(M))_{basic}, d)$

Non-commut. Weil algebra, $\mathcal{W}_{\mathfrak{g}}$

cubic Dirac operator $\mathcal{D}$ (Alekseev-Meinrenken, 2000) (Kostant 1999)

(odd NS element) G

$\downarrow$

$D \longleftarrow \mathcal{D}$ (cubic Dirac)

$d = \{D, \cdot\}$

$$W_{\mathfrak{g}} = \left(S(\mathfrak{g}[\theta] + \mathbb{C}K)/(K=1),\ d = \frac{d}{d\theta} \right) \quad (\theta^2 = 0)$$

odd indeterminate

$\bar{a} = \theta a, \quad [\bar{a}, \bar{b}] = (a, b)K$

$\mathfrak{g}^{super} = \mathfrak{g}[\theta] + \mathbb{C}K$

Quantization $\mathcal{W}_{\mathfrak{g}}$: replace $S(\mathfrak{g}^{super})$ by $U(\mathfrak{g}^{sup})$

Chiralization : replace $\mathfrak{g}^{super}$ by $\hat{\mathfrak{g}}^{super} \longrightarrow V^{k\, super}(\mathfrak{g})$

De Sole – VK 2006

$$\begin{tikzcd}
\mathcal{V}^k_{sup}(\mathfrak g) \arrow[d] & V^k_{sup}(\mathfrak g) \arrow[l] \arrow[d] & L \arrow[d] & G \arrow[d] \\
\text{Weil (complex) algebra } \mathcal{W}(\mathfrak g), \mathcal{D} & \text{non commut. (complex) Weil algebra } W(\mathfrak g), D \arrow[l] & C & D
\end{tikzcd}$$

$$\mathcal{W}(\mathfrak g) = \Big(S\big(\overbrace{\mathfrak g[\theta] + \mathbb{C}K}^{\mathfrak g^{super}}\big)\big/_{K=1}\ ,\ d = \tfrac{d}{d\theta}\Big)$$

where the central extension $\mathfrak g[\theta] + \mathbb{C}K$ is defined by $[\bar a, \bar b] = (a,b)K$, $\{a, b\}$, $\{\bar a, b\}$ as usual, $\bar a = a\theta$

$$W(\mathfrak g) = \Big(U\big(\mathfrak g[\theta] + \mathbb{C}K\big)\big/_{K=1}\ ,\ d = \tfrac{d}{d\theta}\Big)$$

$d = \{\mathcal{D}, \cdot\}$ resp $d = [D, \cdot]$

where $\mathcal{D}$ and D is the cubic Dirac operator (classical and quantum)

$\mathcal{D}^2 = C$ $\qquad D^2 = C + \left(\frac{h^\vee}{24} - \frac{1}{16}\right)\dim \mathfrak g$

H. Cartan ~1950

Alexeev – Meinrenken, Kostant ~ 1999

Explicit formulas:

classical

$$\mathcal{D} = \sum_i a^i \bar{a}^i + \frac{1}{3} \sum_{i,j} \overline{[a^i, a^j]}\, \bar{a}^i \bar{a}^j$$

$$\mathcal{C} = \frac{1}{2} \sum_i a^{i2} + \frac{1}{2} \sum_{i,j} \bar{a}^i [a^i, a^j]\, \bar{a}^j$$

quantum

$$D = \sum_i a^i \bar{a}^i + \frac{1}{3} \sum_{i,j} \overline{[a^i, a^j]}\, \bar{a}^i \bar{a}^j$$

$$C = \frac{1}{2} \sum_i a^{i2} + \frac{1}{2} \sum_{i,j} [a^i, a^j]\, \bar{a}^i \bar{a}^j + \frac{1+2h^\vee}{16} \dim \mathfrak{g}$$

Remarks 1) H. Cartan's theorem for equivariant cohomology:

$$H_G(M) = H\big((W(\mathfrak{g}) \otimes \Omega(M))_{\text{basic}}, d \big)$$

2) $(W(\mathfrak{g}), D)$ is used in representation theory of Lie groups (multiplet decomps, Vogan conjecture)

3) $V^k_{\text{sup}}(\mathfrak{g})$, L, G used in representation theory of the NS algebra. VK-Todorov

Currently in repr. theory of $\hat{\mathfrak{g}}$ VK-Moseneder-Frajria-Papi

Basic idea:

$$\begin{array}{ccc}
\text{ClFT picture} & \xleftarrow{\text{quasicl. lim.}} & \text{QFT picture} \\
 & & \text{replace } \mathcal{U} \text{ by } V^k \\
 & & \Big\uparrow \text{chiralize} \\
\text{Classical-mechanical picture} & \xrightarrow{\text{quantize}} & \text{Quantum-mech picture} \\
\text{e.g. } S(\mathfrak{g} \text{ or } \mathfrak{g}^{\text{super}}) & & \text{replace } S \text{ by } \mathcal{U}
\end{array}$$

More complicated situations reduced to this one if we are able to represent our Poisson algebra

as $PA = H(S(\ldots), d_{cl})$

Then $A = H(\mathcal{U}(\ldots), d_Q)$

$$VA = H(V^k(\ldots), d_{VA})$$

The most important example:

Example 3: W-algebras

ata: $\mathfrak{g}$ simple f.-d. Lie (super) algebra
: nilpotent (even) element of $\mathfrak{g}$, $k \in \mathbb{C}$

nclude: $\mathfrak{g} \supset sl_2 = \langle e, h, f \rangle$, $[e,f]=h$, $[h,e]=e$, $[h,f]=-f$
With resp. to ad h:

$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$, let $\mathfrak{n} = \mathfrak{g}_{\geq \frac{1}{2}} \supset \mathfrak{m} = \mathfrak{g}_{\geq 1}$

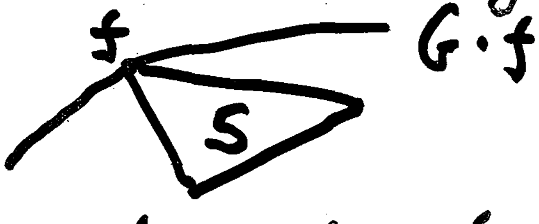
$J = \{ a - (f|a) \mid a \in \mathfrak{m} \}$.

VK-Wakimoto
2003-06

lassical affine W-alg generalized DS reduction 1984) KdV-type equations	Affine W-algebra Represent. theory of Virasoro like algebras
$\mathcal{W}^k(\mathfrak{g}, f)$ $\longleftarrow$	$W^k(\mathfrak{g}, f)$
$\mathcal{W}^{fin}(\mathfrak{g}, f)$ $\longleftarrow$	$W^{fin}(\mathfrak{g}, f)$
$\Vert$	$\Vert$
$(S(\mathfrak{g})/S(\mathfrak{g})J)^{\mathfrak{n}}$	$(U(\mathfrak{g})/U(\mathfrak{g})J)^{\mathfrak{n}}$

Slodowy slice

$S = f + \mathrm{Ker}\, e \in \mathfrak{g}$



For subregular f

$S \longrightarrow \mathfrak{g}/G = \mathfrak{h}/W$

is semiuniversal
deformation of
simple rat. singularity

Quantization of
Slodowy slice

Controls primitive
ideals of $\mathfrak{g}$

Kostant 78, Lynch 79
physicists 90s

Premet, Gan-Ginzburg
2002-09, ..., Losev, 09

True for all f? Yes, if allow only