Virasoro frames and structure codes of the moonshine vertex operator algebra

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Joint works with M. Harada, A. Munemasa
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Virasoro Vertex operator algebra

Let $Vir = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \bigoplus \mathbb{C}\mathbf{c}$ be a Virasoro algebra, i.e.,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}\left(m^3 - m\right)\delta_{m+n,0}\mathbf{c};$$
$$[L_m, \mathbf{c}] = 0.$$

Let $L(c,h)$ be an irreducible highest weight module of Vir of central charge c and highest weight h.

Lemma ([FZ])

$L(c,0)$ is a simple VOA.

When $c = 1/2$, $L\left(\frac{1}{2},0\right)$ is rational and C_2-cofinite.
It has only irreducible modules

$$L\left(\frac{1}{2},0\right), L\left(\frac{1}{2},\frac{1}{2}\right), \text{and } L\left(\frac{1}{2},\frac{1}{16}\right).$$

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Examples. 1. the Moonshine VOA $V^\natural$, ($\mathrm{Aut}\, V^\natural$ =Monster)
2. the lattice VOA $V_{\sqrt{2}A_1}$, and

$$V_{\sqrt{2}A_1} \cong L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{1}{2}, \frac{1}{2})$$

3. Many lattice VOAs and their twisted versions.

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Important facts.

1. All framed VOAs are rational. [DGH]

2. $Stab_V(F) = \{g \in Aut(V) \mid g(F) = F\}$ is finite. [DGH]

3. If V is framed and $V_1 = 0$, then $Aut(V)$ is a finite group.[Miy]

Remark

The subalgebra F is not unique. In general, there are many subalgebra $\cong L(1/2, 0)^{\otimes n}$ in V. In fact, there may be infinitely many.

Problem: Characterizes all Virasoro frames of V (up to the actions of $\mathrm{Aut}(V)$).

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Structure codes

We shall associate two binary codes C and D to V and F.

Let $F \simeq L(1/2, 0)^{\otimes n}$. (rational) Therefore

$$V = \bigoplus_{h_i \in \{0, 1/2, 1/16\}} m_{h_1,\dots,h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n),$$

where $m_{h_1,\dots,h_r}$ is the multiplicity of $L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ in V.

Remark: All the multiplicities are finite and $m_{h_1,\dots,h_r} \le 1$ if all h_i are different from 1/16.

Let $M = L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ be an irreducible module over F. The 1/16-word (or τ-word) $\tau(M)$ of M is a binary codeword $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n$ such that

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$$V = \bigoplus_{h_i \in \{0, 1/2, 1/16\}} m_{h_1,\dots,h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n),$$

where $m_{h_1,\dots,h_r}$ is the multiplicity of $L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ in V.
Remark: All the multiplicities are finite and $m_{h_1,\dots,h_r} \le 1$ if all h_i are different from 1/16.

Let $M = L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ be an irreducible module over F. The 1/16-word (or τ-word) $\tau(M)$ of M is a binary codeword $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n$ such that

$$\beta_i = \begin{cases} 0 & \text{if } h_i = 0 \text{ or } 1/2, \\ 1 & \text{if } h_i = 1/16. \end{cases}$$

For any $\alpha \in \mathbb{Z}_2^n$, denote by V^α the sum of all irreducible submodules M of V such that $\tau(M) = \alpha$.

Lemma

$D := \{\alpha \in \mathbb{Z}_2^n \mid V^\alpha \neq 0\}$ is an even linear subcode of $\mathbb{Z}_2^n$ and $V = \oplus_{\alpha \in D} V^\alpha$. Moreover, V^0 is a subalgebra of V.

For any $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{Z}_2^n$, denote

$$V(\gamma) = L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n),$$

where $h_i = 1/2$ if $\gamma_i = 1$ and $h_i = 0$ elsewhere.

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$C := \{\gamma \in \mathbb{Z}_2^n \mid m_{\gamma_1/2, ..., \gamma_n/2} \neq 0\}$ is an even linear code. Moreover, $V(0) = F$ and $V^0 = \oplus_{\gamma \in C} V(\gamma)$.

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Summarizing, there exists a pair of even linear codes (C, D) such that

$$V = \bigoplus_{\beta \in D} V^{\beta} \quad \text{and} \quad V^0 = \bigoplus_{c \in C} V(c).$$

The codes (C, D) are called the *structure codes* of a framed VOA V associated to the frame F.

Lemma

1. *D is triply even, i.e.,* $\mathrm{wt}(\alpha) \equiv 0 \mod 8$ *for all $\alpha \in D$.*
2. *C is even.*
3. *$D \subset C^{\perp}$.*

Theorem

Let V be a framed VOA with structure codes (C, D). Then, V is holomorphic if and only if $C = D^{\perp}$.

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Lemma ([M96a])

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$$\tau_\beta(u) := (-1)^{\langle \alpha, \beta \rangle} u \quad \text{for } u \in V^\alpha. \tag{1}$$

is an automorphism on V

Note that the subgroup $P = \{\tau_\beta \,|\, \beta \in \mathbb{Z}_2^n\}$ is an elementary abelian 2-group and is isomorphic to the group of irreducible characters $\mathrm{Irr}\, D \cong \mathbb{Z}_2^n / D^\perp$ of D. In addition, the fixed point subspace V^P is equal to $V^0$and all $V^\alpha, \alpha \in D$, are irreducible V^0-modules.

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Similarly, we can define an automorphism on V^0 by

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In this case, the group $Q = \{\sigma_\beta \,|\, \beta \in \mathbb{Z}_2^n\}$ is isomorphic to the dual group $\mathrm{Irr}\, C \cong \mathbb{Z}_2^n / C^\perp$ of C and $(V^0)^Q = T$.

Note that σ_β is just an automorphism of V^0. It DOES NOT necessarily lift to an automorphism of V. Nevertheless, the following holds.

Theorem (Theorem 12 of [LY08])

Let V be a framed VOA with the structure codes (C, D). For a binary word $\xi \in \mathbb{Z}_2^n$, there exists $g \in \mathrm{Aut}(V)$ such that $g|_{V^0} = \sigma_\xi$ if and only if $\xi \cdot \beta \in C$ for all $\beta \in D$, where $\xi \cdot \beta = (\xi_1 \cdot \beta_1, \ldots, \xi_n \cdot \beta_n)$ is the coordinatewise product of ξ and β. Moreover, g has order 2 if $\mathrm{wt}(\xi\cdot\beta) \equiv 0 \mod 4$ for all $\beta \in D$; otherwise, g has order 4.

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Let V be a framed VOA with the structure codes (C, D). For a binary word $\xi \in \mathbb{Z}_2^n$, there exists $g \in \mathrm{Aut}(V)$ such that $g|_{V^0} = \sigma_\xi$ if and only if $\xi \cdot \beta \in C$ for all $\beta \in D$, where $\xi \cdot \beta = (\xi_1 \cdot \beta_1, \ldots, \xi_n \cdot \beta_n)$ is the coordinatewise product of ξ and β. Moreover, g has order 2 if $\mathrm{wt}(\xi \cdot \beta) \equiv 0 \mod 4$ for all $\beta \in D$; otherwise, g has order 4.

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Remark

Since $C = D^{\perp}$ when V is holomorphic, by abuse of notation, we often call the $1/16$-code D a structure code of V, also.

The structure codes C and D are $Aut(V)$-invariant.

Question: Can we classify all structure codes for holomorphic framed VOAs?

In [LY08], it was shown that they satisfy some "duality" conditions.

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Let $V = \bigoplus_{\beta \in D} V^{\beta}$ be a framed VOA with structure codes (C, D). Then, for every non-zero $\beta \in D$, the subcode $C_{\beta} = \{\alpha \in C \mid supp\,\alpha \subset supp\,\beta\}$ contains a doubly even self-dual subcode w.r.t. β.

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Corollary

Let D be a linear binary code of length $16k$, $k \in \mathbb{Z}^+$. Then D can be realized as the $\frac{1}{16}$-code of a holomorphic framed VOA of rank $8k$ if and only if (1) D is triply even and (2) the all-one vector $\mathbf{1} \in D$.

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By the theorem above, the classification of the structure codes for holomorphic framed VOAs is equivalent to the classification of triply even codes of length $16k$.

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Definition

A triply even code of length 48 is called a moonshine code if it can be realized as the $1/16$-code of $V^\natural$.

Proposition

Let D be a moonshine code. Then D satisfies the following conditions:

$$D \text{ is triply even,} \tag{2}$$

$$D \ni \mathbf{1}, \tag{3}$$

$$D^\perp \text{ has minimum weight at least } 4. \tag{4}$$

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1. Determine all triply even codes satisfying the conditions (2) - (4).
2. Determine which one can be realized as a moonshine code.
3. Characterizes all Virasoro frames of $V^\natural$ up to the action of the Monster.

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$\mathbb{Z}_4$-codes and Moonshine codes

Let $\mathcal{C}$ be a self-orthogonal $\mathbb{Z}_4$-code. Define

$$A_4(\mathcal{C}) = \frac{1}{2}\{(x_1,\dots,x_n) \in \mathbb{Z}^n | (x_1,\dots,x_n) \in \mathcal{C} \mod 4\}.$$

Lemma

$A_4(\mathcal{C})$ is even unimodular iff $\mathcal{C}$ is a type II self-dual code.

Note that if $\mathcal{C} = 0$, then $A_4(\mathcal{C}) = (2\mathbb{Z})^n \cong (\sqrt{2}A_1)^n$.
In [DMZ94] and [M98], it was shown that the lattice VOA $V_{\sqrt{2}A_1}$ is framed and

$$V_{\sqrt{2}A_1} \cong L(\frac{1}{2},0) \otimes L(\frac{1}{2},0) \oplus L(\frac{1}{2},\frac{1}{2}) \otimes L(\frac{1}{2},\frac{1}{2})$$

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A Type II $\mathbb{Z}_4$-code $\mathcal{C}$ of length 24 is said to be *extremal* if the minimum Euclidean weight d_E is equal to 16.

Theorem

Let $\mathcal{C}$ be an extremal Type II $\mathbb{Z}_4$-code of length 24. Then

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Structure codes for $V_{A_4(\mathcal{C})}$.

Let $\mathcal{C}$ be a type II self-dual $\mathbb{Z}_4$ code. Denote

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$$\mathcal{C}_1 = \{\alpha \in \{0,1\}^n \,|\, \alpha \equiv \beta \mod 2 \text{ for some } \beta \in \mathcal{C}\}.$$

These codes $\mathcal{C}_0$ and $\mathcal{C}_1$ are called torsion and residue codes, resp.

Lemma

$\mathcal{C}_1$ is doubly even and $\mathcal{C}_0 = \mathcal{C}_1^{\perp}$.

Define three linear maps $d : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$, $\ell : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$ and $r : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$ such that

$$\begin{aligned} d(a_1, a_2, \dots, a_n) &= (a_1, a_1, a_2, a_2, \dots, a_n, a_n), \\ \ell(a_1, a_2, \dots, a_n) &= (a_1, 0, a_2, 0, \dots, a_n, 0), \\ r(a_1, a_2, \dots, a_n) &= (0, a_1, 0, a_2, \dots, 0, a_n), \end{aligned} \qquad (5)$$

for any $(a_1, a_2, \dots, a_n) \in \mathbb{Z}_2^n$.

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Lemma

$\mathcal{C}_1$ is doubly even and $\mathcal{C}_0 = \mathcal{C}_1^{\perp}$.

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Let $\mathcal{C}$ be a self-dual $\mathbb{Z}_4$-code and $\mathcal{C}_0$ and $\mathcal{C}_1$ defined as above. Then the structure codes of the lattice VOA $V_{A_4(\mathcal{C})}$ are given by

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Let θ be an automorphim of $V_{A_4(\mathcal{C})}$, which is a lift of the (-1)-map on the lattice $A_4(\mathcal{C})$.

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Let V^T be the unique irreduicble θ-twisted module for $V_{A_4(\mathcal{C})}$.

Define

$$\tilde{V}_{A_4(\mathcal{C})} = (V_{A_4(\mathcal{C})})^+ \oplus (V^T)^+.$$

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$\tilde{V}_{A_4(\mathcal{C})}$ is a holomorphic framed VOA. Moreover, the structure codes associated to the frame F for $\tilde{V}_{A_4(\mathcal{C})}$ are given by (C^0, D'), where $C^0 = \{\alpha \in C \mid \langle \alpha, (10)^{24} \rangle = 0\}$ and $D' = \langle d(\mathcal{C}_1), (10)^{24} \rangle$.

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Let C be a binary code of length n. We shall define

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to be the code generated by $d(C)$ and $(10)^n$. We call the code $\mathcal{D}(C)$ the *extended doubling (or simply the doubling)* of C.

Lemma

If C is a doubly even $[8n, k]$ code, then the doubling $\mathcal{D}(C)$ is a triply even $[16n, k+1]$ code.

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Let $\mathcal{C}$ be an extremal $\mathbb{Z}_4$-code of length 24. Then the doubling $\mathcal{D}(\mathcal{C}_1)$ of the residue code is a moonshine code.

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By using the sphere packing bound, it is easy to show that $\dim C \leq 41$, $C = D^{\perp}$.

Theorem

Let D be a moonshine code. Then $\dim D \geq 7$.

It was shown in [DGH98] that there exists a moonshine code of dimension 7 and it is isomorphic to

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Theorem (Harada-L-Munemasa)

Up to equivalence, there is a unique binary $[48, 7]$ *code satisfying*

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There is a unique extremal Type II $\mathbb{Z}_4$*-code* $\mathcal{C}$ *of length* 24 *whose residue code has dimension* 6.

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They are equivalent to the doublings $\mathcal{D}(C_{7,1})$, $\mathcal{D}(C_{7,2})$, where $C_{7,1}$, $C_{7,2}$ are doubly even codes of dimension 7.

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Let S be a triply subcode of $\mathcal{D}(C)$ such that $\mathbf{1} \in S$ and the dual weight of S is ≥ 4. Then there exists a subcode C' of C such that S is equivalent to $\mathcal{D}(C')$.

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Lemma

Let V be a holomorphic framed VOA of central charge 24 such that the $1/16$-code of V is a doubling $\mathcal{D}(B)$. Then there exists a type II $\mathbb{Z}_4$-code $\mathcal{C}$ such that $\mathcal{C}_1 = B$ and $\tilde{V}_{A_4(\mathcal{C})} \cong V$.

Theorem

Let B be a binary doubly even code of length 24. Then $\mathcal{D}(B)$ is a moonshine code if and only if there is an extremal Type II $\mathbb{Z}_4$-code $\mathcal{C}$ of length 24 with $B = \mathcal{C}_1$.

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By using computer, we verify that there are two 7-dimensional codes which cannot be realized as the residue codes of any extremal $\mathbb{Z}_4$-codes. Their generator matrices are given by $[\ I_7\ ,\ M_{7,i}\ ]$ $(i = 1, 2)$ where

$$M_{7,1} = \begin{bmatrix} 11101010011111010 \\ 00111100000001110 \\ 00000011000110111 \\ 11111110100000111 \\ 11000010010110001 \\ 01001111111101010 \\ 01011001101100000 \end{bmatrix} \text{ and } M_{7,2} = \begin{bmatrix} 00001111001110000 \\ 11110011110110100 \\ 00101001101101000 \\ 11110000110000010 \\ 11001111000000001 \\ 01111100000111111 \\ 01101001011101111 \end{bmatrix}$$

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In fact, there are many non-realizable doublings.

Table: Numbers of inequivalent codes of length 24 which is doubly even and has dual weight ≥ 4

Dimensions	# codes	# residue codes
12	9	9
11	21	?
10	49	?
9	60	46
8	32	20
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Triply even codes of length 48 have just been classified by Betsumiya-Harada-Munemasa-Shimakura.

Theorem

Let D be a maximal triple even code of length 48. Then D is equivalent to one of the followings.

1. $\mathcal{D}(C)$ *for some doubly even self-dual code of length* 24.
2. $\mathcal{D}(e_8)^{\oplus 3}$, $\mathcal{D}(e_8) \oplus \mathcal{D}(d_{16}^{+})$, *or*
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The minimal weight of $D^{\perp}$ *is* 4 *in Case 1 and 2 and is* 2 *in Case 3.*

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There are exactly 3 codes (up to equivalent) which is minimal subject to (6)–(8)

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3 kinds of Moonshine codes

There are 3 kinds of Moonshine codes.

1. the doublings $\mathcal{D}(C)$ for some doubly even codes.
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Note codes in Case 2 and 3 must be subcodes of $\mathcal{D}(e_8)^{\oplus 3}$ or $\mathcal{D}(e_8) \oplus \mathcal{D}(d_{16}^{+})$.

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Triply even codes satisfying (6) **–** (8)

Dim	15	14	13	12	11	10	9	8	7
T.E. codes	1	4	20	44	86	89	39	7	1
decomposable	1	2	1	1	1	0	0	0	0
doublings	0	0	9	21	49	60	32	7	1
indecomposable but not a doubling	0	2	10	22	36	29	7	0	0

Maximal codes

Lemma

Let C be a doubly even self-dual code of length 24. Then there is an extremal Type II $\mathbb{Z}_4$-code $\mathcal{C}$ with $C = \mathcal{C}_1$.

Hence, we have

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The doubling $\mathcal{D}(C)$ of any doubly even self-dual code C of length 24 is a moonshine code.

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Theorem

Suppose D is a moonshine code. Let $\xi \in \mathbb{Z}_2^{48} \setminus D$ such that $D' = \langle D, \xi \rangle$ is triply even. If $\xi + D$ has the minimal weight 8, then $D' = \langle D, \xi \rangle$ is also a moonshine code.

Corollary

The triply even codes $\mathcal{D}(e_8) \oplus \mathcal{D}(e_8) \oplus \mathcal{D}(e_8)$, $\mathcal{D}(e_8 \oplus e_8) \oplus \mathcal{D}(e_8)$ and $\mathcal{D}(d_{16}) \oplus \mathcal{D}(e_8)$ are moonshine codes.

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Virasoro frames

How about the Virasoro frames of $V^\natural$?

If D is a doubling, then we have a characterization using $\mathbb{Z}_4$-codes.

Theorem (L-Shimakura)

Let $\mathcal{C}$ and $\mathcal{C}'$ be extremal type II $\mathbb{Z}_4$-codes of length 24. Let $V = \tilde{V}_{A_4(\mathcal{C})} \cong V^\natural$. If the Virasoro frame $F(\mathcal{C})$ and $F(\mathcal{C}')$ of V are conjugate under $\mathrm{Aut}(V)(\cong \mathbb{M})$ then $\mathcal{C} \cong \mathcal{C}'$.

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Let $\mathcal{C}$ and $\mathcal{C}'$ be extremal type II $\mathbb{Z}_4$-codes of length 24. Let $V = \tilde{V}_{A_4(\mathcal{C})} \cong V^\natural$. If the Virasoro frame $F(\mathcal{C})$ and $F(\mathcal{C}')$ of V are conjugate under $\mathrm{Aut}(V)(\cong \mathbb{M})$ then $\mathcal{C} \cong \mathcal{C}'$.

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Thank you !

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