

# A complex hyperbolic reflection group and the bimonster

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- Let  $\omega = e^{2\pi i/3}$  and  $p = 2 + \omega$ . Let  $\mathcal{E} = \mathbb{Z}[\omega]$ .
- Let  $R$  be a PID, for example  $R = \mathcal{E}$  (or a ring in quaternions or  $\mathbb{Z}$ ).
- An  *$R$ -lattice*  $L$  is a free  $R$ -module of finite rank with Hermitian form  $L \times L \rightarrow R$  written as  $(x, y) \mapsto \langle x, y \rangle$ .
- *Norm* of  $x \in L$  is  $|x|^2 = \langle x, x \rangle$ .
- Given  $R \subseteq S$ , let  $L^S = L \otimes_R S$ . For example, if  $R = \mathcal{E}$ , then  $L^{\mathbb{C}} = L \otimes_{\mathcal{E}} \mathbb{C}$  is the underlying vector space of  $L$ .
- *dual lattice*:  $L' := \{x \in L^{\text{Frac}(R)} \mid \langle x, L \rangle \subseteq R\}$ .

# Reflection groups

## Definition

$V$  nonsingular inner product space  $/K$ . Let  $r$  be a vector of nonzero norm.

A *reflection*  $\phi_r^\xi$  in  $r$  is an automorphism of  $V$  that fix the hyperplane  $r^\perp$  (called *mirror of reflection*) and multiply  $r$  by a root of unity  $\xi$  in  $K$ .

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$$\phi_r^\xi(x) = x - r(1 - \xi)\langle r, x \rangle / |r|^2 \quad \text{where } \xi \neq 1 \text{ is a root of unity}$$

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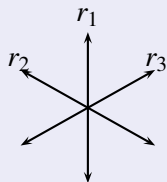
- $r \in L$  is a *root* if  $r$  is primitive and some reflection in  $r$  preserves  $L$ .
- *reflection group* of  $L$  :

$$R(L) = \{\text{gp. generated by the reflections in roots of } L\} \subseteq \text{Aut}(L).$$

# Reflection groups in Lie theory

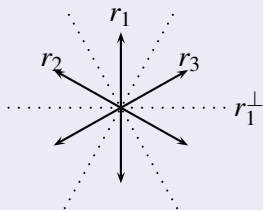
- The eigenvalues (or roots) of the action of maximal abelian lie subalgebra  $\mathfrak{h}$  on the complex simple lie algebra  $\mathfrak{g}$  span a lattice  $L$  called the *root lattice*. The reflection group  $R(L)$  is called the *Weyl group* of  $\mathfrak{g}$ .
- The mirrors of the reflection are hyperplanes in the vector space  $\mathfrak{h}^*$  that partition the space into chambers.
- Let  $\rho$  be the half sum of all the roots on one side of a randomly chosen hyperplane in  $\mathfrak{h}^*$ . It is called the *Weyl vector*.
- Pick the mirrors that are closest to  $\rho$  (in spherical metric).
- the roots perp. to these mirrors are the *simple roots*  $D$  - the vertex set of of the Dynkin diagram.
- The angle at which these mirrors intersect determine the edges on the Dynkin diagram.

# The example of $A_2$



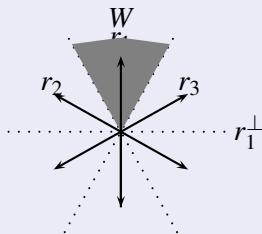
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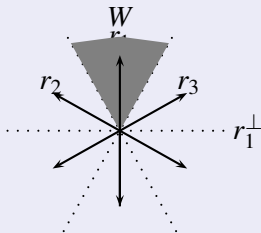
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- $\pm r_1, \pm r_2, \pm r_3$  : roots.
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- $W$  : Weyl chamber
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- $r_i^\perp$  : mirrors.
- $W$  : Weyl chamber
- $r_2, r_3$  : simple roots.

To get the Dynkin diagram:

- Choose a positive half space.
- Define the Weyl vector  $\rho$ :

$$\rho = \text{half sum of positive roots}$$

- Take one root for each of the walls closest to  $\bar{\rho}$ . This is the set of *simple roots*  $D$ .
- Make a graph with vertex set  $D$ : join two vertices  $a$  and  $b$  if the reflections braid ( $aba = bab$ ) and do not join if they commute.
- this gives the Dynkin diagrams in the simply laced cases.

# From diagrams to groups

Let  $D$  be a graph. We define the group  $\text{Cox}(D, n)$ . It has one generator  $g_r$  for each vertex  $r$  of  $D$  and the following relations.

$$\begin{aligned}g_r^n &= 1 \\g_r g_s g_r &= g_s g_r g_s \quad \text{if } (r, s) \text{ is an edge} \\g_r g_s &= g_s g_r \quad \text{otherwise}\end{aligned}$$

- Recall that when  $D$  is the classical A-D-E diagrams  $\text{Cox}(D, 2)$  are presentations for the classical Weyl groups.
- $\text{Cox}(A_{n-1}, \infty)$  is the  $n$  strand braid group.

# two Presentations of the bimonster

Theorem [Conway-Norton-Ivanov-Simons. '90s]

Let  $D$  be the incidence graph of the projective plane over  $\mathbb{F}_3$  and  $M_{666} \subseteq D$  be the subgraph with 16 vertices looking like an  $Y$  with 5 vertices on each hand. Then the bimonster has the following two presentation:

$$\begin{array}{ccc} \text{Cox}(M_{666}, 2) & \xrightarrow{\quad\quad\quad} & (M \times M) \rtimes \mathbb{Z}/2 \\ & \searrow & \nearrow \text{deflate} \\ & \text{Cox}(D, 2) & \end{array}$$

The kernel of the map from  $\text{Cox}(M_{666}, 2)$  is generated by  $sp^{10} = 1$  where  $sp = ab_1c_1ab_2c_2ab_3c_3$ . The kernel of the map from  $\text{Cox}(D, 2)$  is generated by “deflating” all the free 12-gons inside  $D$ .

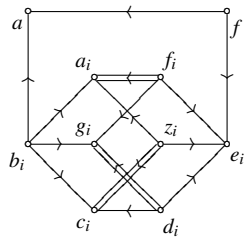
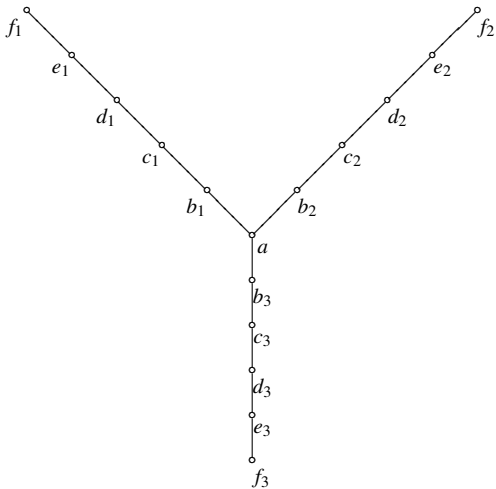
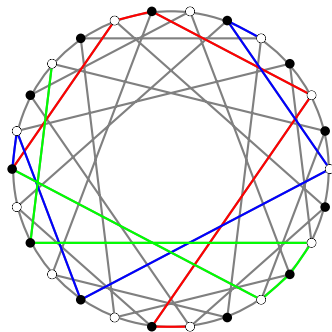
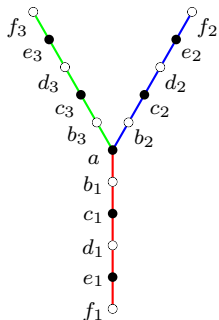


Figure: The  $M_{666}$  and  $\text{Inc}(\mathbb{P}^2(\mathbb{F}_3))$  diagram

# The diagrams



The  $M_{666}$  graph and the incidence graph of  $\mathbb{P}^2(\mathbb{F}_3)$ .

# The $M_{666}$ diagram in reflection group of a lattice $L$

Let  $E_8^{\mathcal{E}}$  = complex  $E_8$  defined / $\mathcal{E}$ .

Let  $H = \mathcal{E}e_1 \oplus \mathcal{E}e_2$  with

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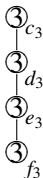
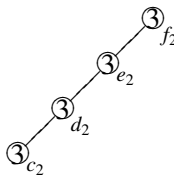
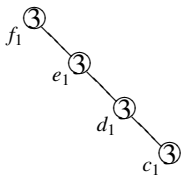
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Let  $L = 3E_8^{\mathcal{E}} \oplus H$ . Since

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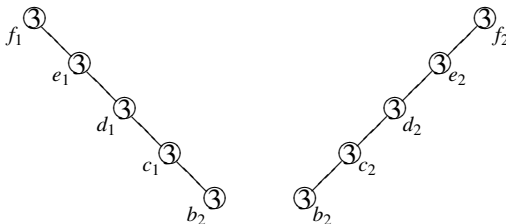
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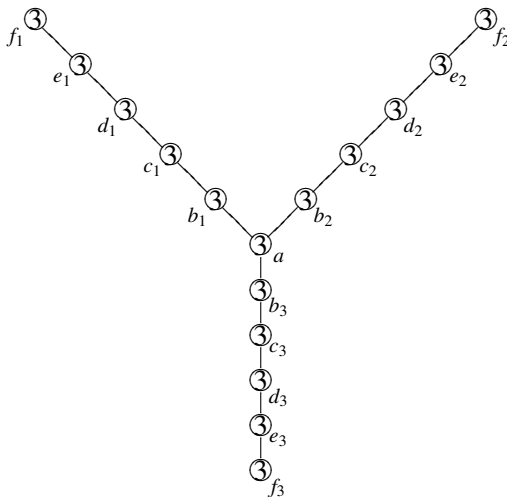
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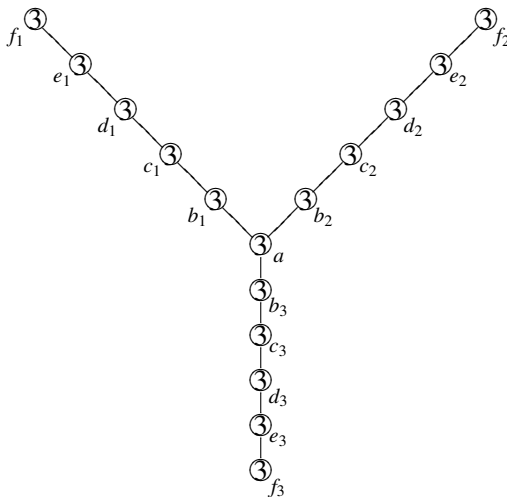
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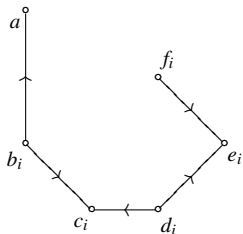
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- “affinize” to get three  $A_5$ .
- “hyperbolize” to get  $M_{666}$ .
- two nodes  $x$  and  $y$  are joined  $\iff$  the reflections braid  $\iff |\langle x, y \rangle| = \sqrt{3}$
- not joined  $\iff$  reflections commute  $\iff \langle x, y \rangle = 0$ .



# The 26 node diagram in reflection group of $L$

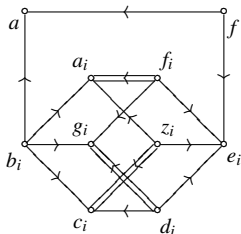
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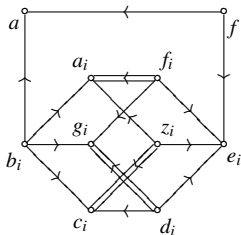
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- More precisely we have found 26 roots (vector of norm  $-3$ ) in  $L$  labelled by vertices of  $D$ . Call these the *simple roots*.
- order three reflections in the simple roots braid/commute according to the graph  $D$ .

## Theorem 1 [B. '05]

Let  $L = 3E_8^{\mathcal{E}} \oplus H$ . (In-fact  $L$  is the unique  $\mathcal{E}$  lattice of signature  $(1, 13)$  such that  $pL' = L$ .) Then  $\text{Cox}(D, 3) \twoheadrightarrow \text{Aut}(L)$ .

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**Next:** Realize  $D$  as a crude substitute of a Dynkin diagram for the reflection group of  $L$ .

# Another picture of the incidence graph of $\mathbb{P}^2(\mathbb{F}_3)$ .

	$f$	$b_1$	$f_3$	$b_2$	$z_1$	$f_1$	$d_2$	$d_1$	$d_3$	$b_3$	$f_2$	$z_2$	$z_3$
$a$	○	○		○						○			
$a_1$		○	○		○						○		
$a_2$			○	○		○						○	
$c_2$				○	○		○						○
$e_1$	○				○	○		○					
$g_1$		○				○	○		○				
$g_3$			○				○	○		○			
$g_2$				○				○	○		○		
$c_3$					○				○	○		○	
$a_3$						○				○	○		○
$e_2$	○						○				○	○	
$c_1$		○						○				○	○
$e_3$	○		○						○				○

# Constructing the lattice $L$ from the diagram $D$ (Contd.)

Let  $x_1, \dots, x_{13}$  be the 13 row vectors in  $\mathcal{E}^{1,13}$ , given below.

1	-1	-1		-1						-1			
1		-1	-1	-1	-1					-1			
1			-1	-1	-1	-1					-1		
1	-1				-1	-1	-1						-1
1		-1				-1	-1	-1					
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Then  $L = \{c \in \mathcal{E}^{1,13} : \langle x_i, c \rangle \equiv 0 \pmod{p}\}$ .

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From now on we write vectors in  $L$  in this co-ordinate system. For example

- The vectors  $x_1, \dots, x_{13}$  given above correspond to the thirteen points of  $\mathbb{P}^2(\mathbb{F}_3)$ .
- the co-ordinates for the 13 lines  $l_1, \dots, l_{13}$  are  $(0|p, 0, \dots, 0), (0|0, p, \dots, 0), \dots, (0|0, \dots, 0, p)$ .

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- Clearly the group  $L_3(3)$  acts on  $L$ .
- $L_3(3)$  fixes a two dimensional primitive sublattice  $F \subseteq L$  generated by  $w_{\mathcal{L}} = (p|0, 0, \dots, 0)$  and  $w_{\mathcal{P}} = (4| -1, -1, \dots, -1)$ .
- There exists another diagram automorphism  $\sigma \in \text{Aut}(L)$  of order 12 that corresponds to interchanging points and lines.
- The group  $\langle \sigma, L_3(3) \rangle / \mathcal{E}^*$  acts faithfully on the 13 dimensional complex hyperbolic space  $\mathbb{C}H^{13} = \mathbb{P}^+(L^{\mathbb{C}})$ .

# the Weyl vector

- Let  $\xi = e^{-\pi i/6}$  and  $(\rho_1, \dots, \rho_{26}) = (x_1, \dots, x_{13}, \xi l_1, \dots, \xi l_{13})$ . Let  $\bar{\rho} = \frac{1}{26} \sum_{i=1}^{26} \rho_i$ .
- The 13 mirrors  $x_i^\perp$  meet at  $[w_{\mathcal{P}}]$ .
- The 13 mirrors  $l_i^\perp$  meet at  $[w_{\mathcal{L}}]$ .
- The automorphism  $\sigma$  interchanges  $[w_{\mathcal{P}}]$  and  $[w_{\mathcal{L}}]$ .
- $[\bar{\rho}]$  is the midpoint of the geodesic joining  $[w_{\mathcal{P}}]$  and  $[w_{\mathcal{L}}]$ . It is the Unique point in  $\mathbb{P}^+(L^{\mathbb{C}})$  fixed by  $\text{Aut}(D)$ . Define  $ht(r) = \langle \bar{\rho}, r \rangle / |\bar{\rho}|^2$ .

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## Theorem 2 [B. '05]

The simple roots have height 1 and all other roots have strictly larger height. In other words the 26 simple mirrors  $x_i^\perp$  and  $l_i^\perp$  are the mirrors closest to  $\bar{\rho}$ .

# Proof of theorem 1: simple reflections generate $R(L)$

## Proof.

**Step I:** Write down finitely many reflections generating  $R(L)$ .

This uses ideas in Allcock's earlier work where he showed  $\text{Aut}(L)/R(L) \twoheadrightarrow 6$ . Suz, namely use special properties of the Leech lattice and the Heisenberg gp. in  $R(L)$ .

**Step II:** Run a height reduction algorithm (computer aided): Take a root  $r$  obtained in step 1. Reflect it in the simple roots to decrease distance from  $\bar{\rho}$  and repeat until you reach a simple root. (perturb if you get stuck). Repeat with each of the roots obtained in step 1. □

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  - ▶ However they do not lead to the Weyl vector in the natural way.
- Another important discrepancy is that reflection in vectors of  $D$  does not always reduce height - but it almost always does.

# Connection with the bimonster

- $\text{Cox}(D, \infty)$  maps onto both  $\text{Aut}(L)$  and  $M \wr 2$  with  $2.L_3(3)$  as automorphisms.
- There are analogs in  $\text{Aut}(L)$  of the extra relators needed for presenting the bimonster :
  - ▶ the spider relator have order 20 (instead of 10).
  - ▶ The deflation relations are also true in  $\text{Aut}(L)$ . (Corollary of this:  $\text{Aut}(L)$  can be generated by the 14 reflections of the  $M_{556}$  diagram).

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## Conjecture [Allcock '97]

Let  $X = (\mathbb{P}^+(L^{\mathbb{C}}) \setminus \{\text{mirrors of } R(L)\}) / \mathbb{P} \text{Aut}(L)$ . Then  $\pi_1(X) \rightarrow M \wr 2$ .

# Connection with the bimonster

- $\text{Cox}(D, \infty)$  maps onto both  $\text{Aut}(L)$  and  $M \wr 2$  with  $2.L_3(3)$  as automorphisms.
- There are analogs in  $\text{Aut}(L)$  of the extra relators needed for presenting the bimonster :
  - ▶ the spider relator have order 20 (instead of 10).
  - ▶ The deflation relations are also true in  $\text{Aut}(L)$ . (Corollary of this:  $\text{Aut}(L)$  can be generated by the 14 reflections of the  $M_{556}$  diagram).

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Possible ramifications:

- A moduli interpretation of  $X$ .
- Connection with the conjectural Monster Manifold.

# Fundamental group of a space modulo a group action

Let  $H$  be a group acting discontinuously on a space  $B$ . Fix a base point  $b \in B$ . Let  $\pi_1(B//H)$  consist of all pairs  $(\gamma, t)$ , where  $t \in H$  and  $\gamma$  is a homotopy class of paths in  $B$  from  $b$  to  $tb$ . Define the multiplication on  $\pi_1(B//H)$  by

$$(\gamma, t) * (\gamma', t') = (\gamma * t\gamma', tt').$$

Define  $\pi_H : \pi_1(B//H) \rightarrow H$  by  $\pi_H(\gamma, t) = t$ . The kernel of this surjection is  $\pi_1(Y^\circ, \bar{\rho})$ . So we have an exact sequence

$$1 \rightarrow \pi_1(B, b) \rightarrow \pi_1(B//H) \xrightarrow{\pi_H} H \rightarrow 1$$

# The fundamental group of the mirror-complement-quotient

- Recall:  $X = (\mathbb{P}^+(L^{\mathbb{C}}) \setminus \{\text{mirrors}\}) / \mathbb{P} \text{Aut}(L)$ .  $\pi_1(X)$  is defined in the orbifold sense.
- Choose basepoint at  $\bar{\rho}$ .
- Given a simple root  $v \in D$  let  $\gamma_v$  be the path which starts at  $\bar{\rho}$ , follows the geodesic from  $\bar{\rho}$  to  $v^{\perp}$  till very close to  $v^{\perp}$ , rotates  $120^\circ$  in the plane perpendicular to  $v^{\perp}$  and then follows the geodesic upto  $\phi_v(\bar{\rho})$ . Let  $g_v = (\gamma_v, \phi_v) \in \pi_1(X)$ .

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## Theorem [B. '07]

Let  $v, v' \in D$  be simple roots. If the reflections  $\phi_v$  and  $\phi_{v'}$  braid/commute in  $R(L)$  then the loops  $g_v$  and  $g_{v'}$  braid/commute in  $\pi_1(X)$ .

*idea of proof:* Construct explicit homotopies between paths in  $\mathbb{P}^+(L^{\mathbb{C}}) \setminus \{\text{mirrors}\}$  using the information about the location of mirrors around  $\bar{\rho}$  provided by theorem 2.

# a toy example of homotopy

- Let  $\tilde{\gamma}_v$  be the geodesic joining  $\bar{\rho}$  and  $\phi_v(\bar{\rho})$ .
- Let  $C$  be the triangle between  $\gamma_v$  and  $\tilde{\gamma}_v$ .
- Since there are no mirrors closer to  $\bar{\rho}$  than  $v^\perp$  there are no mirrors intersecting the interior of  $C$ .
- So  $\gamma_v$  and  $\tilde{\gamma}_v$  are homotopic in  $\mathbb{P}^+(L^\mathbb{C}) \setminus \{\text{mirrors}\}$  and so  $(\gamma_v, \phi_v) = (\tilde{\gamma}_v, \phi_v)$  in  $X$ .

# Results analogous to Allcock's conjecture: real reflection groups

## Theorem [Brieskorn, Saito, Deligne]

Let  $K$  be a root lattice/ $\mathbb{Z}$  for the Weyl group  $R(K)$  with Dynkin diagram  $\Delta$ . Then the braid space  $X = (K^{\mathbb{C}} \setminus \{\text{mirrors of } R(K)\})/R(K)$  is a  $K(\pi, 1)$  and  $\pi_1(X) = \text{Cox}(\Delta, \infty)$ .

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**Remark:** These spaces sometimes have moduli interpretation. For example consider the symmetric group  $S^n = W(A_{n-1})$  acting on  $\mathbb{C}^n$ . Then  $X = (\mathbb{C}^n \setminus \{x_i \neq x_j\})/S^n = \{\text{space of } n \text{ distinct points in } \mathbb{C}\}$ .  
and

$\pi_1(X) = \text{Cox}(A_{n-1}, \infty) = \text{braid group on } n \text{ strands}$ .

# Moduli interpretation in the complex case

## Theorem [Allcock, Carleson, Toledo]

Let  $L = \mathcal{E}^{1,4}$ . Then  $(\mathbb{P}^+(L^{\mathbb{C}}) \setminus \{\text{mirrors}\})/R(L)$  is the space of smooth cubic surfaces in  $\mathbb{P}_{\mathbb{C}}^3$ .

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## Theorem [Deligne, Mostow]

Let  $L = E_8^{\mathcal{E}} \oplus E_8^{\mathcal{E}} \oplus H$ . Then  $(\mathbb{P}^+(L^{\mathbb{C}}) \setminus \{\text{mirrors}\})/\text{Aut}(L)$  is the space of twelve distinct unmarked points in  $\mathbb{P}_{\mathbb{C}}^1$ .

The mirrors correspond to degenerations, i.e singular surfaces or points collapsing on each other. Infact the second example above is the largest discrete arithmetic subgroup of  $U(1, n)$  constructed in the Deligne - Mostow articles.

# More on the reflection group of $L = E_8^{\mathcal{E}} \oplus E_8^{\mathcal{E}} \oplus H$

One has analogs of theorem 1 and 2 for  $L = E_8^{\mathcal{E}} \oplus E_8^{\mathcal{E}} \oplus H$ .

## Theorem

- 1.[Allcock] One has  $\text{Cox}(\tilde{A}_{11}, 3) \twoheadrightarrow R(L)$ .
- 2.[B. '06] The action of the dihedral group  $D_{12}$  on  $\tilde{A}_{11}$  extends to an action on  $L$  and hence  $\mathbb{P}^+(L^{\mathbb{C}})$ . There is a unique point  $\bar{\rho}$  fixed by the diagram automorphisms  $D_{12}$  and the twelve simple mirrors corresponding to the affine  $A_{11}$  diagram are the only ones closest to  $\bar{\rho}$ .

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- In the moduli interpretation  $\bar{\rho}$  corresponds to the configuration of 12 points arranged around the meridian of  $\mathbb{P}_{\mathbb{C}}^1$ .
- The twelve closest mirrors correspond to two of the adjacent points coming together.
- Let  $v \in \tilde{A}_{11}$ . The loop  $\gamma_v$  around the simple mirror  $v^{\perp}$  correspond to interchanging two adjacent points. This explains the validity of the braid/commute relations in  $\pi_1(X)$ .

# A parallel story for the quaternions

- Replace the ring  $\mathcal{E} = \mathbb{Z}[e^{2\pi i/3}]$  by the ring of Hurwitz integers  $\mathcal{H}$  consisting of  $(a + bi + cj + dk)/2$ , with  $a, b, c, d$  integers congruent modulo 2.
- Replace the incidence graph of  $\mathbb{P}^2(\mathbb{F}_3)$  by  $\mathbb{P}^2(\mathbb{F}_2)$ .
- Let  $L = \text{Leech}^{\mathcal{H}} \oplus H = 3E_8^{\mathcal{H}} \oplus H$ .
- Then there are analogs of theorem 1 and 2 for  $L$ .
- There are 14 quaternionic reflections of order 4 in  $R(L)$  that form the diagram  $\text{Inc}\mathbb{P}^2(\mathbb{F}_2)$  and they generate  $R(L)$ .
- This 14 node diagram is obtained by extending the  $M_{444}$  diagram.
- We see that there is a unique “Weyl vector” in the quaternionic hyperbolic space that is fixed by the diagram automorphisms.
- The 14 simple mirrors are the mirrors closest to the Weyl vector.

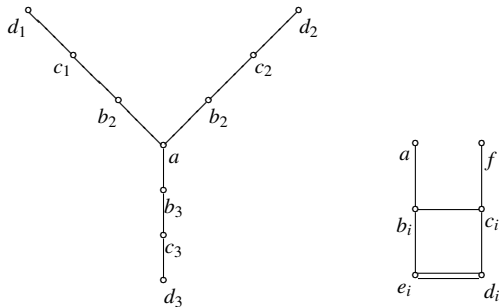


Figure: The  $M_{444}$  and  $Inc(\mathbb{P}^2(\mathbb{F}_2))$  diagram

# From functions invariant under $\text{Aut}(L)$ to modular forms

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Start with a modular form with poles at cusps (e.g.  $1/\Delta$ ). Take Borcherds Singular theta lift to get an  $O(2, 26)$  form. Then restrict to  $\mathbb{C}H^{13} \subseteq Gr^+(2, 26)$  to get an meromorphic form on  $\bar{X}$  invariant under  $\text{Aut}(L)$  with zeroes along the mirrors.

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In fact it is easy to write down meromorphic automorphic forms invariant under  $\text{Aut}(L)$  directly. Take

$$E_n(z) = \sum_{r \in \Phi} \langle r, z \rangle^{-6n}$$

Where  $\Phi$  is the set of roots of  $L$ .

These also also meromorphic for  $n$  large enough, invariant under  $\text{Aut}(L)$ , and have poles along the mirrors.

# Identifying the $L_3(3)$ fixed points with upper half plane

Recall  $F =$  the 2 dimensional sublattice of  $L$  fixed by  $L_3(3)$  pointwise.

## Proposition

There exists an isometry  $\beta : \mathbb{P}^+(F^{\mathbb{C}}) \rightarrow \mathcal{H}_{\mathbb{R}}^2$  such that  $\beta(\bar{\rho}) = i$ ,  $\beta(w_{\mathcal{P}}) = p$ ,  $\beta(w_{\mathcal{L}}) = -p^{-1}$  and  $\beta(w_{\mathcal{P}} - \bar{\omega}pw_{\mathcal{L}}) = 0$ . The diagram automorphism  $\sigma$  acts on the upper half plane by  $\tau \mapsto -1/\tau$ . Let  $\Gamma_F = \{g \in \mathbb{P} \text{Aut}(L) : gF = F\}$ . Given  $\gamma \in \text{SL}_2(\mathbb{Z})$ ,  $\beta^{-1}\gamma\beta$  extends to an automorphism of  $L$  if and only if  $\gamma$  belongs to a conjugate of  $\Gamma_0(13)$ .

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So restricting the Borcherds forms or the  $E_n(z)$ 's to  $\mathbb{P}^+(F^{\mathbb{C}})$  gives modular forms for a genus zero group (the restrictions are non-trivial as  $\mathbb{P}^+(F^{\mathbb{C}})$  is not contained in any mirrors).

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