Vector-valued modular forms and modular differential equations

Edinburgh, September 2009

”Differential equations satisfied by modular functions have been studied since the time of Jacobi.”

Harnad & McKay, Mason, Gaberdiel, Zhu, ...

Connection with Schwarzian, Picard-Fuchs, etc. equations, integrable systems, string theory, topological field theory, ...

RCFT and VOA theory: vector-valued modular forms (genus one characters/trace functions) satisfying modular differential equations.

Reflect the structure of the operator algebra (null vectors).

Scalar modular forms

Holomorphic functions $f\colon \mathbf{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z})$ and some (even positive integer) w.

$f(\tau)$ is periodic $\Rightarrow$ q-expansion (with $q = e^{2\pi i \tau}$)

$$f = \sum_{n \in \mathbb{Z}} f_n q^n$$

Classically: holomorphicity at $\tau = i\infty$.

More generally: finite order pole.

Example: Eisenstein series (for $k > 1$)

$$E_{2k}(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k\text{-}1}(n)\, q^n$$

where

$$\sigma_k(n) = \sum_{d|n} d^k$$

and B_k is the k-th Bernoulli number.

$$E_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k}\, E_{2k}(\tau)$$

Any holomorphic modular form is a polynomial in E_4 and E_6.

Examples: $E_8 = E_4^2$ and $E_{10} = E_4 E_6$.

Discriminant form (weight 12)

$$\Delta = \frac{1}{1728}\left(E_4^3 - E_6^2\right) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

Doesn't vanish on $\mathbf{H}$!

Hauptmodul (weight 0)

$$J(q) = \frac{E_4^3}{\Delta} - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots$$

Univalent map $J : X(1) \rightarrow \mathbb{CP}^1$.

Any modular form of weight 0 is a polynomial in $J(q)$.

Applications:

- number theory, algebraic geometry, combinatorics, ...

- algebra (VOA-s, affine Lie-algebras, Moonshine)

- physics (string theory and 2d CFT)

Need for a theory of vector-valued modular forms!

Eichler, Gross-Zagier, Knopp-Mason, Borcherds, Eholzer-Skoruppa, ...

- forms for (finite index) subgroups, Jacobi forms, ...

Vector-valued modular forms

Holomorphic maps $\mathbb{X}:\mathbf{H}\rightarrow\mathbb{C}^d$ such that

$$\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^w\,\rho\begin{pmatrix}a\,b\\c\,d\end{pmatrix}\mathbb{X}(\tau)$$

ρ is a suitable (projective) representation of $\mathrm{SL}_2(\mathbb{Z})$.

Exponent matrix: $\exp(2\pi\mathrm{i}\boldsymbol{\Lambda})=\rho\begin{pmatrix}1\,1\\0\,1\end{pmatrix}$.

q-expansion:

$$\exp(-2\pi\mathrm{i}\boldsymbol{\Lambda}\tau)\,\mathbb{X}(\tau)=\sum_{n\in\mathbb{Z}}\mathbb{X}[n]\,q^n$$

$\mathbb{X}$ meromorphic at $\tau=\mathrm{i}\infty$ if singular part $\mathcal{P}\mathbb{X}=\sum_{n<0}\mathbb{X}[n]\,q^n$ is a finite sum.

Thanks to Δ, general case may be reduced to $w=0$, in which case ρ is a true representation of $\mathrm{PSL}_2(\mathbb{Z})$.

$\mathcal{M}(\rho)$: linear space of weight 0 forms.

Singular part map $\mathcal{P}:\mathcal{M}(\rho)\to\mathbb{C}[q^{-1}]^d$ is affected by choice of $\mathbf{\Lambda}$.

Is there a "best" choice?

Trace condition: $\mathcal{P}$ is bijective if and only if

$$\boxed{\mathrm{Tr}(\mathbf{\Lambda}) = d - \frac{\alpha}{2} - \frac{\beta_1 + 2\beta_2}{3}}$$

$d, \alpha, \beta_1, \beta_2$: non-negative integers (signature of ρ).

The fundamental matrix

Multiplication by $J(q)$ takes $\mathcal{M}(\rho)$ to itself $\Rightarrow$

$\mathcal{M}(\rho)$ is a $\mathbb{C}[J]$-module (of finite rank)

$\mathcal{P}$ is bijective $\Leftrightarrow$ there exists a unique matrix $\boldsymbol{\Xi}(q)$ (the fundamental matrix) such that

1. the columns of $\Xi(q)$ generate $\mathcal{M}(\rho)$;

2. the limit

$$\mathcal{X} = \lim_{q \to 0} \left(q^{-\boldsymbol{\Lambda}} \boldsymbol{\Xi}(q) - q^{-1} \right)$$

exists (characteristic matrix).

Invertibility of $\Xi(q)$ & transformation rule

$$\boldsymbol{\Xi}\Big(\frac{a\tau+b}{c\tau+d}\Big) = \rho\begin{pmatrix} a\,b \\ c\,d \end{pmatrix} \boldsymbol{\Xi}(\tau)$$

$\Rightarrow$ each component of $\mathcal{J}\mathbb{X} = \Xi(q)^{-1}\mathbb{X}(q)$ is a polynomial in $J(q)$.

Alternative characterizations of $\mathbb{X}$:

1. polynomial representation $\mathcal{J}\mathbb{X} = \Xi(q)^{-1}\,\mathbb{X}(q)$;
2. q-expansion $\mathbb{X}(q) = \sum_{n\in\mathbb{Z}} \mathbb{X}[n]\,q^{\boldsymbol{\Lambda}+n}$;
3. singular part $\mathcal{P}\mathbb{X} = \sum_{n<0} \mathbb{X}[n]\,q^{n}$.

The inversion formula

Relates polynomial representation to singular part.

$$\mathcal{J}\mathbb{X}(w) = \frac{1}{2\pi \mathrm{i}} \oint \frac{J'(q)}{w - J(q)} \Xi(q)^{-1} \, q^{\mathbf{\Lambda}} \mathcal{P}\mathbb{X}(q) \, dq$$

Polynomial representation reduces questions to elementary algebra (univariate polynomials).

Should we know $\Xi(q)$, we would know the structure of $\mathcal{M}(\rho)$.

How can we determine $\Xi(q)$?

Differential operators

Modular covariant derivative (of weight w)

$$\mathsf{D}_w = \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\tau} - \frac{w}{12}E_2(\tau)$$

takes a vector-valued modular form of weight w (and multiplier ρ) into a form of weight $w+2$ with the same multiplier $\Rightarrow$

$$\mathsf{D}^n = \mathsf{D}_{w+2(n-1)} \circ \cdots \circ \mathsf{D}_{w+2} \circ \mathsf{D}_w$$

increases the weight by $2n$.

n	1	2	3	4	5	6
$\mathfrak{E}_n(\tau)$	$\frac{E_{10}}{\Delta}$	$\frac{E_8}{\Delta}$	$\frac{E_6}{\Delta}$	$\frac{E_4}{\Delta}$	$\frac{E_{14}}{\Delta^2}$	$\frac{1}{\Delta}$

and $\mathfrak{E}_{n+6}(\tau) = \dfrac{\mathfrak{E}_n(\tau)}{\Delta(\tau)}$

The operators

$$\nabla_n = \mathfrak{E}_n(\tau)\,\mathsf{D}^n$$

(with $\nabla_0 = \mathbf{1}$) map each $\mathcal{M}(\rho)$ to itself: they are *universal*.

Any universal differential operator is a combination of the $\boldsymbol{\nabla}_n$-s, with coefficients in $\mathbb{C}[J]$.

The product of two universal operators is again universal.

$$\nabla_1 \circ \nabla_1 = (\mathbf{J} - 984) \circ \nabla_2 - \tfrac{1}{6}(5\mathbf{J} + 264) \circ \nabla_1$$
$$\nabla_1 \circ \nabla_2 = (\mathbf{J} + 744) \circ \nabla_3 - \tfrac{2}{3}(\mathbf{J} - 984) \circ \nabla_2$$
$$\nabla_n \circ \nabla_6 = \nabla_{n+6}$$

with $\mathbf{J} : \mathbb{X}(\tau) \mapsto J(\tau)\,\mathbb{X}(\tau)$ the multiplication-by-J operator.

The algebra $\mathbb{D}=\mathbb{C}[\mathbf{J},\nabla_1,\nabla_2,\ldots]$ is independent of ρ and w.

$\mathbb{D}$ is generated by $\mathbf{J},\nabla_1,\nabla_2$ and ∇_3.

$$\begin{aligned}
1728\nabla_4 &= \nabla_2\circ\nabla_2-\nabla_1\circ\nabla_3+\tfrac{5}{6}(\mathbf{J}+744)\circ\nabla_3+\tfrac{1}{9}(5\mathbf{J}+264)\circ\nabla_2\\
\nabla_5 &= \nabla_2\circ\nabla_3+(\mathbf{J}+744)\circ\nabla_4-\tfrac{1}{3}(\mathbf{J}+744)\circ\nabla_3\\
\nabla_6 &= \nabla_2\circ\nabla_4-\nabla_3\circ\nabla_3-\tfrac{5}{6}\nabla_5+\tfrac{1}{6}(5\mathbf{J}-6648)\circ\nabla_4\\
&\quad -\tfrac{1}{18}(5\mathbf{J}+264)\circ\nabla_3
\end{aligned}$$

Each $\mathcal{M}(\rho)$ is a $\mathbb{D}$-module

Vector-valued modular forms $\leftrightsquigarrow$ representations of $\mathbb{D}$.

The differential equations

∇_n maps $\mathcal{M}(\rho)$ to itself $\Rightarrow$ the matrix entries of

$$\mathcal{D}_n(q) = \Xi(q)^{-1}\,\nabla_n \Xi(q)$$

are polynomials in $J(q)$.

$$\mathcal{D}_1(q) = (J(q) - \mathcal{X})\,(\mathbf{\Lambda} - 1) + \mathbf{\Lambda}\mathcal{X} - 240\,(\mathbf{\Lambda} - 1)$$
$$\mathcal{D}_2(q) = (J(q) - \mathcal{X})(\mathbf{\Lambda} - 1)(\mathbf{\Lambda} - \tfrac{7}{6}) + (\mathbf{\Lambda} - \tfrac{1}{6})\mathbf{\Lambda}\mathcal{X} + 504(\mathbf{\Lambda} - 1)(\mathbf{\Lambda} - \tfrac{73}{63})$$

and

$$\mathcal{D}_3(q) = (J(q) - \mathcal{X})\,(\mathbf{\Lambda} - 1)(\mathbf{\Lambda} - \tfrac{7}{6})(\mathbf{\Lambda} - \tfrac{4}{3}) + (\mathbf{\Lambda} - \tfrac{1}{3})(\mathbf{\Lambda} - \tfrac{1}{6})\mathbf{\Lambda}\mathcal{X}$$
$$- 480\,(\mathbf{\Lambda} - 1)\,(\mathbf{\Lambda}^2 - \tfrac{101}{40} + \tfrac{71}{45})$$

Fundamental matrix satisfies the ODEs

$$\boxed{\nabla_n \Xi(q) = \Xi(q)\,\mathcal{D}_n(q)}$$

The spectral condition

First order (Fuchsian!) ODE

$$\boxed{\nabla_1 \Xi(q) = \Xi(q)\,\mathcal{D}_1(q)}$$

with boundary condition

$$\mathcal{X} = \lim_{q\to 0}\left(q^{-\boldsymbol{\Lambda}}\Xi(q) - q^{-1}\right)$$

May be solved for any $\boldsymbol{\Lambda}$ and $\mathcal{X}$ (e.g. recursively).

Higher order equations $\Rightarrow$ spectral condition

$$(J-984)\,\{\mathcal{D}_2+(J+744)\,(\boldsymbol{\Lambda}-1)\}=\left\{\mathcal{D}_1+\tfrac{1}{6}\,(5J+264)\right\}\mathcal{D}_1$$

$$(J+744)\left\{\mathcal{D}_3+(J-984)\,(\boldsymbol{\Lambda}-1)\left(\boldsymbol{\Lambda}-\tfrac{7}{6}\right)\right\}=\left\{\mathcal{D}_1+\tfrac{2}{3}\,(J-984)\right\}\mathcal{D}_2$$

The matrices

$$\mathcal{A} = \tfrac{31}{36}\,(1-\mathbf{\Lambda}) - \tfrac{1}{864}\,(\mathcal{X}+\mathbf{\Lambda}\mathcal{X}-\mathcal{X}\mathbf{\Lambda})$$

and

$$\mathcal{B} = \tfrac{41}{24}\,(1-\mathbf{\Lambda}) + \tfrac{1}{576}\,(\mathcal{X}+\mathbf{\Lambda}\mathcal{X}-\mathcal{X}\mathbf{\Lambda})$$

satisfy the monodromy equation

$$\boxed{\mathcal{A}(\mathcal{A}-1)=\mathcal{B}(\mathcal{B}-1)(\mathcal{B}-2)=0}$$

For a given $\mathbf{\Lambda}$: system of quadratic equations for $\mathcal{X}$.

Generalized Rademacher-Petersson formula

$$\mathcal{X}_{ij} = 2\pi\sqrt{\frac{1-\mathbf{\Lambda}_{jj}}{\mathbf{\Lambda}_{ii}}}\sum_{n=1}^{\infty}\frac{1}{n}I_1\left(\frac{4\pi}{n}\sqrt{\left(1-\mathbf{\Lambda}_{jj}\right)\mathbf{\Lambda}_{ii}}\right)\mathcal{S}_{ij}(n)$$

Example: the Ising model

$$\mathbf{\Lambda} = \frac{1}{48}\begin{pmatrix} 47 & & \\ & 23 & \\ & & 2 \end{pmatrix} \qquad \rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

$$q^{-\mathbf{\Lambda}}\mathbf{\Xi}(q) = q^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2325 & 94208 \\ 1 & 275 & -4096 \\ 1 & -25 & -23 \end{pmatrix} +$$

$$+\, q\begin{pmatrix} 1 & 60630 & 9515008 \\ 1 & 13250 & -1130496 \\ 1 & -4121 & 253 \end{pmatrix} + q^2\begin{pmatrix} 1 & 811950 & 356765696 \\ 1 & 235500 & -63401984 \\ 1 & -102425 & -1794 \end{pmatrix} + \dots$$

Can be summed up

$$\Xi(q) = \begin{pmatrix} \frac{\mathfrak{f}+\mathfrak{f}_1}{2} & \frac{\mathfrak{f}^{25}-\mathfrak{f}_1^{25}-25\mathfrak{f}-25\mathfrak{f}_1}{2} & 8\left(\mathfrak{f}^{17}\mathfrak{f}_1^{8}-\mathfrak{f}^{24}\mathfrak{f}_1-16\mathfrak{f}\right)+\frac{\mathfrak{f}_2^7}{\sqrt{2}}\left(\mathfrak{f}^{39}-\mathfrak{f}_1^{39}-16\mathfrak{f}^{15}-32\mathfrak{f}_1^{15}\right) \\ \frac{\mathfrak{f}-\mathfrak{f}_1}{2} & \frac{\mathfrak{f}^{25}+\mathfrak{f}_1^{25}-25\mathfrak{f}+25\mathfrak{f}_1}{2} & 8\left(\mathfrak{f}^{17}\mathfrak{f}_1^{8}+\mathfrak{f}^{24}\mathfrak{f}_1-16\mathfrak{f}\right)-\frac{\mathfrak{f}_2^7}{\sqrt{2}}\left(\mathfrak{f}^{39}+\mathfrak{f}_1^{39}-16\mathfrak{f}^{15}+32\mathfrak{f}_1^{15}\right) \\ \frac{\mathfrak{f}_2}{\sqrt{2}} & -\left(25+\mathfrak{f}_2^{24}\right)\frac{\mathfrak{f}_2}{\sqrt{2}} & \mathfrak{f}^{15}\mathfrak{f}_1^{7}\left(\mathfrak{f}^{24}-16\right)-16\mathfrak{f}^{24}\frac{\mathfrak{f}_2}{\sqrt{2}} \end{pmatrix}$$

in terms of the classical Weber functions

$$\mathfrak{f}(q) = q^{-\frac{1}{48}} \prod_{n=0}^{\infty} \left(1+q^{n+\frac{1}{2}}\right)$$

$$\mathfrak{f}_1(q) = q^{-\frac{1}{48}} \prod_{n=0}^{\infty} \left(1-q^{n+\frac{1}{2}}\right)$$

$$\mathfrak{f}_2(q) = \sqrt{2}q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1+q^n)$$

MODE-s

$$\left.\begin{array}{r}\text{characters of RCFT}\\ \text{trace functions of VOA}\end{array}\right\} \Rightarrow \text{modular differential equations}$$

$$\mathsf{D}^n\mathbb{X} + \sum_{k=1}^{n} g_k(\tau)\,\mathsf{D}^{n-k}\mathbb{X} = 0$$

with g_k a (weakly) holomorphic modular form of weight $2k$.

Relation to null vectors: Zhu, Gaberdiel-Keller, ...

Needs to determine the annihilator

$$\mathrm{Ann}_{\mathbb{D}}(\mathbb{X}) = \{\nabla \in \mathbb{D} \,|\, \nabla\mathbb{X} = 0\}$$

The annihilator

$\nabla_n \mathbb{X} \in \mathcal{M}(\rho) \Rightarrow \boxed{\partial_n \mathbb{X} := \mathfrak{J}(\nabla_n \mathbb{X})} = \Xi(q)^{-1}\, \nabla_n \mathbb{X} \in \mathbb{C}[J]^d.$

Recursion formula

$$\partial_{n+1}\mathbb{X} = \frac{1}{\mathfrak{a}_n}\left\{\mathcal{D}_1(J) + \mathfrak{b}_n - (J-984)\,(J+744)\,\frac{\mathrm{d}}{\mathrm{d}J}\right\}\partial_n \mathbb{X}$$

where

$$\left.\begin{aligned} \mathfrak{a}_n &= \frac{\mathfrak{E}_1(\tau)\,\mathfrak{E}_n(\tau)}{\mathfrak{E}_{n+1}(\tau)} \\ \mathfrak{b}_n &= \mathfrak{E}_1(\tau)\,\mathfrak{E}_n(\tau)\,\mathrm{D}_n\left(\frac{1}{\mathfrak{E}_n(\tau)}\right) \end{aligned}\right\} \in \mathbb{C}[J]$$

$n \pmod 6$	0	1	2	3	4	5
$\mathfrak{a}_n$	1	$J-984$	$J+744$	$J-984$	1	$(J-984)(J+744)$
$\mathfrak{b}_n$	0	$\frac{5}{6}(J+264)$	$\frac{2}{3}(J-984)$	$\frac{1}{2}(J+744)$	$\frac{1}{3}(J-984)$	$\frac{1}{6}(7J-1704)$

$\mathcal{F}_n =$ submodule of $\mathbb{C}[J]^d$ generated by $\partial_0\mathbb{X}, \partial_1\mathbb{X}, \ldots, \partial_n\mathbb{X}$.

$\mathbb{C}[J]^d$ is Noetherian $\Rightarrow$ the sequence $\mathcal{F}_0 < \mathcal{F}_1 < \ldots$ saturates.

$\exists N$ such that $\mathcal{F}_n = \mathcal{F}_N$ for $n > N$; in particular, $\partial_n\mathbb{X} \in \mathcal{F}_N$, i.e.

$$\partial_n\mathbb{X} = \sum_{k=0}^{N} \mathfrak{p}_k^{(n)}(J)\, \partial_k\mathbb{X}$$

for some polynomials $\mathfrak{p}_k^{(n)} \in \mathbb{C}[J] \Rightarrow$

$$\boxed{\mathfrak{D}_n = \nabla_n - \sum_{k=0}^{N} \mathfrak{p}_k^{(n)}(J)\, \nabla_k \in \mathsf{Ann}_{\mathbb{D}}(\mathbb{X})}$$

To each element of the module

$$\mathsf{Syz}(\mathbb{X}) = \left\{ \begin{pmatrix} \mathfrak{s}_0 \\ \vdots \\ \mathfrak{s}_N \end{pmatrix} \in \mathbb{C}[J]^{N+1} \;\middle|\; \sum_{k=0}^{N} \mathfrak{s}_k(J)\, \partial_k \mathbb{X} = 0 \right\}$$

of syzygys of $\mathcal{F}_N$ corresponds an element

$$\sum_{k=0}^{N} \mathfrak{s}_k(J)\, \nabla_k \in \mathsf{Ann}_{\mathbb{D}}(\mathbb{X})$$

If $\mathfrak{s}^{(1)}, \ldots, \mathfrak{s}^{(r)} \in \mathsf{Syz}(\mathbb{X})$ freely generate $\mathsf{Syz}(\mathbb{X})$, let

$$\mathfrak{S}_m = \sum_{k=0}^{N} \mathfrak{s}_k^{(m)}(J)\, \nabla_k$$

The operators $\mathfrak{D}_n$ and $\mathfrak{S}_m$ generate $\mathsf{Ann}_{\mathbb{D}}(\mathbb{X})$ as a $\mathbb{C}[\mathbf{J}]$-module

Holomorphic MODE-s

$$\sum_{k=0}^{n} g_{n-k}(\tau)\,\mathsf{D}^k \in \mathsf{Ann}^{\mathtt{hol}}(\mathbb{X})$$

if each $g_k(\tau)$ is a holomorphic modular form of weight $2k$.

$$\mathsf{Ann}^{\mathtt{hol}}(\mathbb{X}) = \bigoplus_{n=1}^{\infty} \mathsf{Ann}^{\mathtt{hol}}(\mathbb{X})_n$$

a direct sum of finite dimensional linear spaces.

Hilbert-Poincaré series:

$$H_{\mathbb{X}}(z) = \sum_{n=1}^{\infty} z^n \dim \mathsf{Ann}^{\mathtt{hol}}(\mathbb{X})_n$$

A basis of $\mathsf{Ann}^{\mathtt{hol}}(\mathbb{X})_n$ may be computed systematically using techniques similar to those just discussed.

Ising MODE-s

$$\mathbb{X}(\tau) = \frac{1}{2}\begin{pmatrix} \mathfrak{f}(\tau) + \mathfrak{f}_1(\tau) \\ \mathfrak{f}(\tau) - \mathfrak{f}_1(\tau) \\ \sqrt{2}\mathfrak{f}_2(\tau) \end{pmatrix} \quad \text{and} \quad \mathcal{J}\mathbb{X} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

n	0	1	2	3
$\partial_n\mathbb{X}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\frac{1}{48}\begin{pmatrix} 240-J \\ 24 \\ 3 \end{pmatrix}$	$\frac{1}{768}\begin{pmatrix} 3J+1448 \\ 112 \\ -7 \end{pmatrix}$	$\frac{1}{36864}\begin{pmatrix} 23648-51J \\ 856 \\ 107 \end{pmatrix}$

Saturation index $N = 2$.

$$55296\,\partial_3\mathbb{X} = (22632 - 23J)\,\partial_0\mathbb{X} + 2568\,\partial_1\mathbb{X}$$

$$110592\,\partial_4\mathbb{X} = (17112 + 23J)\,\partial_0\mathbb{X} - 1758\,\partial_1\mathbb{X} + 5136\,\partial_2\mathbb{X}$$

Generators of $\mathsf{Ann}_{\mathbb{D}}(\mathbb{X})$:

$$\mathfrak{D}_3 = \nabla_3 - \frac{107}{2304}\nabla_1 + \frac{23}{55296}(\mathbf{J} - 984)$$
$$\mathfrak{D}_4 = \nabla_4 - \frac{107}{2304}\nabla_2 + \frac{293}{18432}\nabla_1 - \frac{23}{110592}(\mathbf{J} + 744)$$
$$\vdots$$

$\mathsf{Syz}(\mathbb{X})$ trivial $\Rightarrow$ no syzygy operators $\mathfrak{S}_{\mathfrak{m}}$.

Holomorphic operators:

$$\mathsf{D}^3 - \frac{107}{2304}E_4(\tau)\,\mathsf{D} + \frac{23}{55296}E_6(\tau)$$
$$\mathsf{D}^4 - \frac{107}{2304}E_4(\tau)\,\mathsf{D}^2 + \frac{293}{18432}E_6(\tau)\,\mathsf{D} - \frac{23}{110592}E_8(\tau)$$

Hilbert-Poincaré series:

$$H(z) = z^3 + z^4 + 2z^5 + 3z^6 + \cdots = \frac{z^3}{(1-z)(1-z^2)(1-z^3)} + O\left(z^{25}\right)$$

The Haagerup subfactor

Subfactors of von Neumann algebras: ADE type classification if Jones-index < 4.

Haagerup: finite-depth, irreducible, hyperfinite subfactor with smallest index $\frac{5+\sqrt{13}}{2} \approx 4.3 > 4$.

Believed *not* to come from RCFT, VOA, quantum groups, ...

Exponent and characteristic matrix may be determined (Gannon).

If RCFT origin, then character vector with non-negative integer coefficients.

Possible character vector

$$q^{-\Lambda}\mathbb{X} = \begin{pmatrix} q^{-2} + 6q^{-1} + 197085 + 22677397q \\ 80q^{-1} + 1331 + 15762727q \\ 81q^{-1} + 1296 + 15957810q \\ 82q^{-1} + 1544 + 15961934q \\ 3q^{-1} + 240 + 593325q \\ 27q^{-1} + 567 + 5321241q \\ 119q^{-1} + 1623 + 23442192q \\ 42q^{-1} + 777 + 8276275q \\ 5q^{-1} + 229 + 987158q \\ 14q^{-1} + 441 + 2760821q \\ 7q^{-1} + 292 + 1381392q \\ 13q^{-1} + 347 + 2563296q \end{pmatrix} + O\left(q^2\right)$$

Saturation index $N = 12$, two syzygys.

$$H_{\mathbb{X}}(z) = z^{23} + 3z^{24} + 4z^{25} + 9z^{26} + 11z^{27} + 13z^{28} + \cdots$$

Outlook

- vector-valued modular forms pretty well under control
- explicit computational procedures
- representation theory of $\mathbb{D}$

and open questions

- automorphic forms for other groups
- Hecke operators
- Jacobi and Siegel forms