Computing Excluded Minors

Stephan Kreutzer

University of Oxford

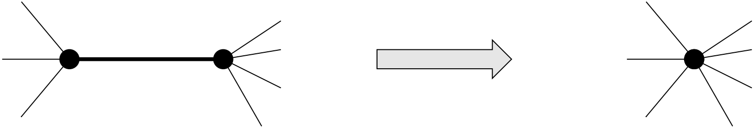
joint work with
Isolde Adler, Bruno Courcelle and Martin Grohe

Logic and Algorithms
July 25, 2008

Minors

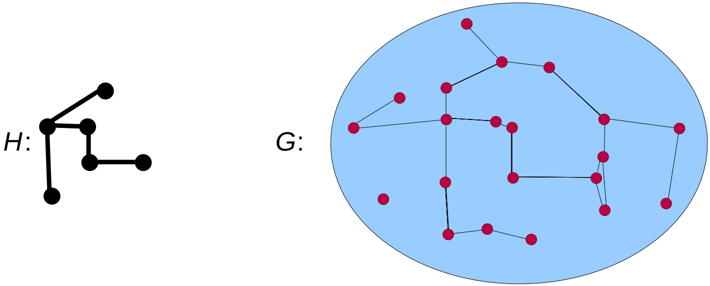
Definition:

A minor of a graph G is a graph obtained from a sub-graph of G by contracting edges.



Notation: We write $H \preceq G$ if H is a minor of G

Minors



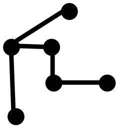
Minor Map: Function $\rho : V(H) \cup E(H) \longrightarrow \{G' : G' \subseteq G\}$ such that

- for each $v \in V(H)$,
  - $\rho(v) \subseteq G$ is a connected subgraphso that
  - $u \neq v$ implies $\rho(u) \cap \rho(v) = \emptyset$
- for each $e = \{u, v\} \in E(H)$,

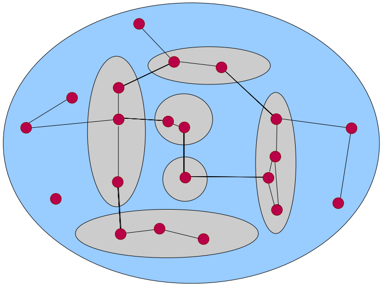
  $\rho(e) := \{u', v'\} \in E(G)$ so that $u' \in \rho(u)$ and $v' \in \rho(v)$.

Minors

H:



G:

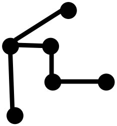


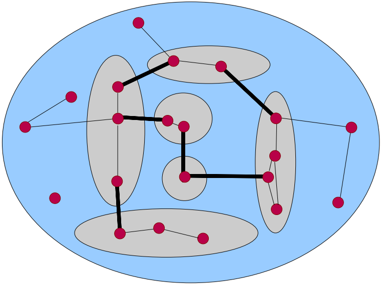
Minor Map: Function $\rho : V(H) \cup E(H) \longrightarrow \{G' : G' \subseteq G\}$ such that

- for each $v \in V(H)$,
  - $\rho(v) \subseteq G$ is a connected subgraphso that
  - $u \neq v$ implies $\rho(u) \cap \rho(v) = \emptyset$
- for each $e = \{u, v\} \in E(H)$,

  $\rho(e) := \{u', v'\} \in E(G)$ so that $u' \in \rho(u)$ and $v' \in \rho(v)$.

Minors

H: 

G: 

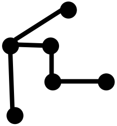
Minor Map: Function $\rho : V(H) \cup E(H) \longrightarrow \{G' : G' \subseteq G\}$ such that

- for each $v \in V(H)$,
  - $\rho(v) \subseteq G$ is a connected subgraphso that
  - $u \neq v$ implies $\rho(u) \cap \rho(v) = \emptyset$
- for each $e = \{u, v\} \in E(H)$,

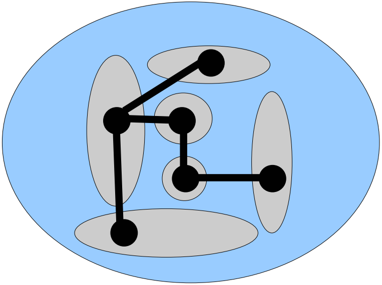
  $\rho(e) := \{u', v'\} \in E(G)$ so that $u' \in \rho(u)$ and $v' \in \rho(v)$.

Minors

H:



G:



Minor Map: Function $\rho : V(H) \cup E(H) \longrightarrow \{G' : G' \subseteq G\}$ such that

- for each $v \in V(H)$,
  - $\rho(v) \subseteq G$ is a connected subgraphso that
  - $u \neq v$ implies $\rho(u) \cap \rho(v) = \emptyset$
- for each $e = \{u, v\} \in E(H)$,

  $\rho(e) := \{u', v'\} \in E(G)$ so that $u' \in \rho(u)$ and $v' \in \rho(v)$.

Minor Ideals

Minor ideals.

Class $\mathcal{C}$ of graphs is a minor ideal if it is closed under taking minors.

A minor ideal $\mathcal{C}$ is proper if it not the class of all (finite) graphs.

Excluded Minors.

H is an excluded minor of a class $\mathcal{C}$ if $H \not\preceq G$ for all $G \in \mathcal{C}$.

Definition. $\mathcal{H}$ class of graphs

$\textsc{Excl}(\mathcal{H}) := \{G : H \not\preceq G \text{ for all } H \in \mathcal{H}\}$

Minor ideals $\mathcal{C}$ can be characterised by sets of excluded minors.

E.g. $\mathcal{C} = \textsc{Excl}\big(\textsc{Graphs} \setminus \mathcal{C}\big)$

Minor Ideals

Minor ideals.

Class $\mathcal{C}$ of graphs is a minor ideal if it is closed under taking minors.

A minor ideal $\mathcal{C}$ is proper if it not the class of all (finite) graphs.

Excluded Minors.

H is an excluded minor of a class $\mathcal{C}$ if $H \not\preceq G$ for all $G \in \mathcal{C}$.

Definition. $\mathcal{H}$ class of graphs

$\textsc{Excl}(\mathcal{H}) := \{G : H \not\preceq G \text{ for all } H \in \mathcal{H}\}$

Minor ideals $\mathcal{C}$ can be characterised by sets of excluded minors.

E.g. $$\mathcal{C} = \textsc{Excl}\Big(\textsc{Graphs} \setminus \mathcal{C}\Big)$$

Examples of Minor Ideals

Examples of Minor Ideals

Forests.

The class $\mathcal{F}$ of forests excludes the triangle.

Planar graphs.

Theorem. (Kuratowski 1930, Wagner 1937)

A graph is planar iff it excludes K_5 and $K_{3,3}$.

Surfaces. Let S be a surface.

The class of graphs that can be drawn on S is a minor ideal.

Examples of Minor Ideals

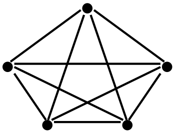
Forests.

The class $\mathcal{F}$ of forests excludes the triangle.

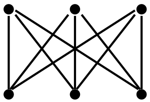
Planar graphs.

Theorem. (Kuratowski 1930, Wagner 1937)

A graph is planar iff it excludes K_5 and $K_{3,3}$.



K_5



$K_{3,3}$

Surfaces. Let S be a surface.

The class of graphs that can be drawn on S is a minor ideal.

Examples of Minor Ideals

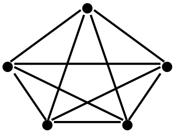
Forests.

The class $\mathcal{F}$ of forests excludes the triangle.

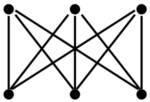
Planar graphs.

Theorem. (Kuratowski 1930, Wagner 1937)

A graph is planar iff it excludes K_5 and $K_{3,3}$.



K_5



$K_{3,3}$

Surfaces. Let S be a surface.

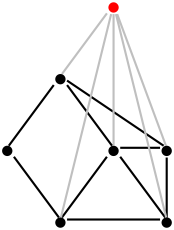
The class of graphs that can be drawn on S is a minor ideal.

Examples of Minor Ideals

Apex graphs.

Graphs from which we can delete one vertex to make it planar.

The class of apex graphs is a proper minor ideal.



Examples of Minor Ideals

Distance from planarity.

A graph G has distance k from planarity if we can remove $\leq k$ vertices to make it planar.

For each $k \geq 0$, the class of graphs of distance k from planarity is a proper minor ideal.

More generally.

If $\mathcal{C}$ is a minor ideal, define distance k from $\mathcal{C}$.

For each $k \geq 0$, the class of graphs of distance k from $\mathcal{C}$ is a proper minor ideal.

A non-example. The class of graphs of crossing number k, for some fixed $k \geq 1$, is not minor closed.

Examples of Minor Ideals

Distance from planarity.

A graph G has distance k from planarity if we can remove $\leq k$ vertices to make it planar.

For each $k \geq 0$, the class of graphs of distance k from planarity is a proper minor ideal.

More generally.

If $\mathcal{C}$ is a minor ideal, define distance k from $\mathcal{C}$.

For each $k \geq 0$, the class of graphs of distance k from $\mathcal{C}$ is a proper minor ideal.

A non-example. The class of graphs of crossing number k, for some fixed $k \geq 1$, is not minor closed.

Examples of Minor Ideals

Distance from planarity.

A graph G has distance k from planarity if we can remove $\leq k$ vertices to make it planar.

For each $k \geq 0$, the class of graphs of distance k from planarity is a proper minor ideal.

More generally.

If $\mathcal{C}$ is a minor ideal, define distance k from $\mathcal{C}$.

For each $k \geq 0$, the class of graphs of distance k from $\mathcal{C}$ is a proper minor ideal.

A non-example. The class of graphs of crossing number k, for some fixed $k \geq 1$, is not minor closed.

Tree-Width of Graphs

The tree-width of a graph measures its similarity to a tree.

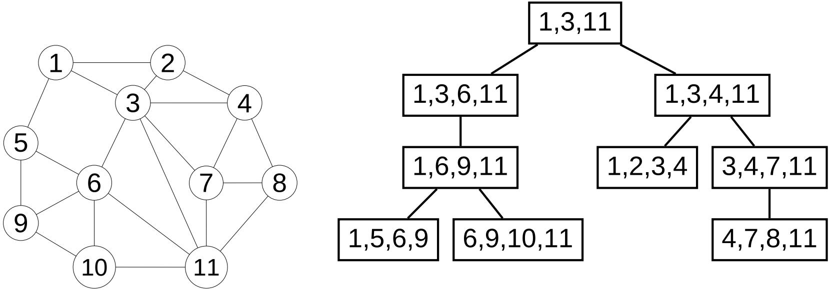
Tree-decomposition of G.

Tree whose nodes are labelled by bags, sets of vertices, such that

- every edge $e \in E(G)$ is completely contained in a bag
- for every vertex $v \in V(G)$ the set of bags containing v induces a subtree

Width. Width of a decomposition is maximum order of a bag -1

Tree-width of a graph is the minimum width of its tree-decompositions.



Tree-Width of Graphs

The tree-width of a graph measures its similarity to a tree.

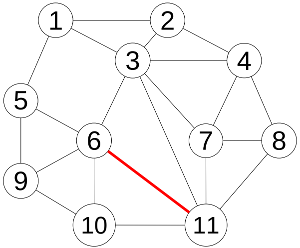
Tree-decomposition of G.

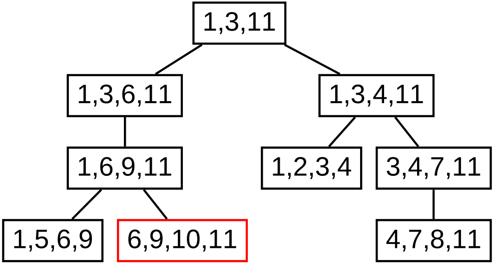
Tree whose nodes are labelled by bags, sets of vertices, such that

- every edge $e \in E(G)$ is completely contained in a bag
- for every vertex $v \in V(G)$ the set of bags containing v induces a subtree

Width. Width of a decomposition is maximum order of a bag -1

Tree-width of a graph is the minimum width of its tree-decompositions.





Tree-Width of Graphs

The tree-width of a graph measures its similarity to a tree.

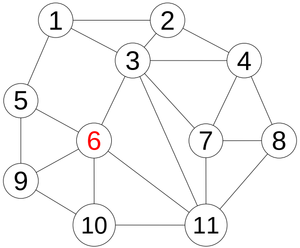
Tree-decomposition of G.

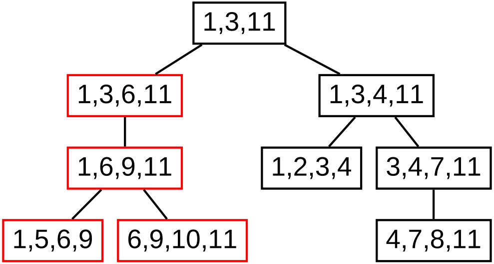
Tree whose nodes are labelled by bags, sets of vertices, such that

- every edge $e \in E(G)$ is completely contained in a bag
- for every vertex $v \in V(G)$ the set of bags containing v induces a subtree

Width. Width of a decomposition is maximum order of a bag -1

Tree-width of a graph is the minimum width of its tree-decompositions.





Tree-Width of Graphs

The tree-width of a graph measures its similarity to a tree.

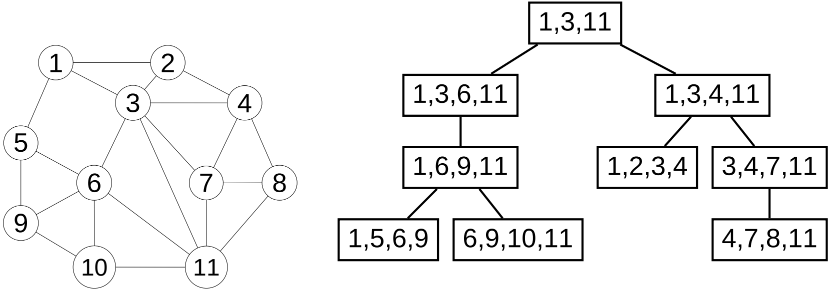
Tree-decomposition of G.

Tree whose nodes are labelled by bags, sets of vertices, such that

- every edge $e \in E(G)$ is completely contained in a bag
- for every vertex $v \in V(G)$ the set of bags containing v induces a subtree

Width. Width of a decomposition is maximum order of a bag -1

Tree-width of a graph is the minimum width of its tree-decompositions.



Examples of Minor Ideals

Theorem. For every $k \geq 1$ the class $\mathcal{T}_k$ of graphs of tree-width $\leq k$ is a proper minor ideal.

Theorem. For every $k \geq 1$ the class $\mathcal{B}_k$ of graphs of branch-width $\leq k$ is a proper minor ideal.

Further Minor Ideals.

- The class of linklessly embeddable graphs
- The class of knotlessly embeddable graphs

Examples of Minor Ideals

Theorem. For every $k \geq 1$ the class $\mathcal{T}_k$ of graphs of tree-width $\leq k$ is a proper minor ideal.

Theorem. For every $k \geq 1$ the class $\mathcal{B}_k$ of graphs of branch-width $\leq k$ is a proper minor ideal.

Further Minor Ideals.

- The class of linklessly embeddable graphs
- The class of knotlessly embeddable graphs

Examples of Minor Ideals

Theorem. For every $k \geq 1$ the class $\mathcal{T}_k$ of graphs of tree-width $\leq k$ is a proper minor ideal.

Theorem. For every $k \geq 1$ the class $\mathcal{B}_k$ of graphs of branch-width $\leq k$ is a proper minor ideal.

Further Minor Ideals.

- The class of linklessly embeddable graphs
- The class of knotlessly embeddable graphs

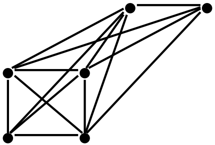
Examples of Minor Ideals

Theorem. For every $k \geq 1$ the class $\mathcal{T}_k$ of graphs of tree-width $\leq k$ is a proper minor ideal.

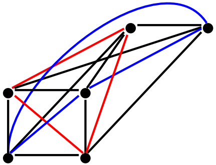
Theorem. For every $k \geq 1$ the class $\mathcal{B}_k$ of graphs of branch-width $\leq k$ is a proper minor ideal.

Further Minor Ideals.

- The class of linklessly embeddable graphs
- The class of knotlessly embeddable graphs



Linkless embedding of K_6 - one edge



Two linked cycles in K_6

Characterising Minor Ideals by Excluded Minors

The Graph Minor Theorem

Definition. $\mathcal{H}$ class of graphs

$\text{EXCL}(\mathcal{H}) := \{G : H \not\preceq G \text{ for all } H \in \mathcal{H}\}$

Clearly: every proper minor ideal $\mathcal{C}$ excludes at least one minor.

E.g. $\mathcal{C} \quad = \quad \text{EXCL}\Big(\text{GRAPHS} \setminus \mathcal{C}\Big)$

Theorem. (Robertson, Seymour)

Every minor ideal $\mathcal{C}$ is characterised by a finite set of obstructions, i.e.

there is $m \geq 0$ and graphs $H_1, \ldots, H_m$ such that

$$\mathcal{C} \quad = \quad \{G : H_i \not\preceq G \text{ for all } i \leq m\}$$

The Graph Minor Theorem

Definition. $\mathcal{H}$ class of graphs

$\text{EXCL}(\mathcal{H}) := \{G : H \not\preceq G \text{ for all } H \in \mathcal{H}\}$

Clearly: every proper minor ideal $\mathcal{C}$ excludes at least one minor.

E.g. $\mathcal{C} = \text{EXCL}\Big(\text{GRAPHS} \setminus \mathcal{C}\Big)$

Theorem. (Robertson, Seymour)

Every minor ideal $\mathcal{C}$ is characterised by a finite set of obstructions, i.e.

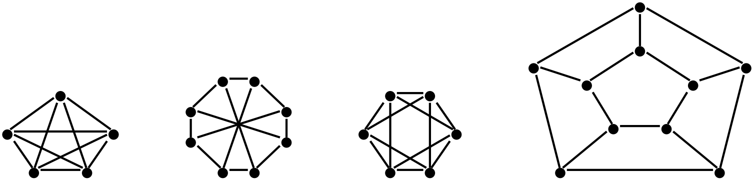
there is $m \geq 0$ and graphs $H_1, \ldots, H_m$ such that

$$\mathcal{C} = \{G : H_i \not\preceq G \text{ for all } i \leq m\}$$

Known Minor Characterisations

Tree-Width.

- Tree-width 1: K_3
- Tree-width 2: K_4
- Tree-width 3: the following 4 graphs



- Tree-width ≥ 4: unknown

Linklessly and knotlessly embeddable graphs.

- Linkless. 7 graphs including K_6
- Knotless. unknown

Minor Ideals

Theorem. (Robertson, Seymour)

Every minor ideal $\mathcal{C}$ is characterised by a finite set of obstructions, i.e.

there is $m \geq 0$ and graphs $H_1, \dots, H_m$ such that

$$\mathcal{C} = \{G : H_i \not\preceq G \text{ for all } i \leq m\}$$

Theorem. For every fixed H, testing whether $H \preceq G$ can be done in cubic time.

Consequence. Every minor ideal has a cubic time membership test.

$\rightsquigarrow$ knotlessly embeddable graphs can be decided in cubic time.

But how? Nobody knows the excluded minors.

Question. Can we compute excluded minor characterisations?

Minor Ideals

Theorem. (Robertson, Seymour)

Every minor ideal $\mathcal{C}$ is characterised by a finite set of obstructions, i.e.

there is $m \geq 0$ and graphs $H_1, \ldots, H_m$ such that

$$\mathcal{C} \quad = \quad \{G : H_i \not\preceq G \text{ for all } i \leq m\}$$

Theorem. For every fixed H, testing whether $H \preceq G$ can be done in cubic time.

Consequence. Every minor ideal has a cubic time membership test.

$\rightsquigarrow$ knotlessly embeddable graphs can be decided in cubic time.

But how? Nobody knows the excluded minors.

Question. Can we compute excluded minor characterisations?

Minor Ideals

Theorem. (Robertson, Seymour)

Every minor ideal $\mathcal{C}$ is characterised by a finite set of obstructions, i.e.

there is $m \geq 0$ and graphs $H_1, \dots, H_m$ such that

$$\mathcal{C} \quad = \quad \{G : H_i \not\preceq G \text{ for all } i \leq m\}$$

Theorem. For every fixed H, testing whether $H \preceq G$ can be done in cubic time.

Consequence. Every minor ideal has a cubic time membership test.

$\rightsquigarrow$ knotlessly embeddable graphs can be decided in cubic time.

But how? Nobody knows the excluded minors.

Question. Can we compute excluded minor characterisations?

Computing Excluded Minors

Obstruction Sets

Let $\mathcal{C}$ be a proper minor ideal.

Definition.

H is an obstruction for $\mathcal{C}$ if

- $H \not\preceq G$ for all $G \in \mathcal{C}$ but
- $H' \in \mathcal{C}$ for all $H' \preceq H$ with $H' \neq H$.

Definition. $\mathcal{O}(\mathcal{C})$ class of obstructions of $\mathcal{C}$

Observation.

- The class of obstructions is unique (up to isomorphism).
- The class of obstructions is the minimal minor characterisation.

Graph Minor Theorem. (Robertson, Seymour)

$\mathcal{O}(\mathcal{C})$ is finite for all minor ideals.

Obstruction Sets

Let $\mathcal{C}$ be a proper minor ideal.

Definition.

H is an obstruction for $\mathcal{C}$ if

- $H \not\preceq G$ for all $G \in \mathcal{C}$ but
- $H' \in \mathcal{C}$ for all $H' \preceq H$ with $H' \neq H$.

Definition. $\mathcal{O}(\mathcal{C})$ class of obstructions of $\mathcal{C}$

Observation.

- The class of obstructions is unique (up to isomorphism).
- The class of obstructions is the minimal minor characterisation.

Graph Minor Theorem. (Robertson, Seymour)

$\mathcal{O}(\mathcal{C})$ is finite for all minor ideals.

A Negative Result

Theorem. (Fellows, Langston 1989)

The following problem is undecidable:

Given: Turing machine M deciding a minor ideal $\mathcal{C}$
Problem: Compute an excluded minor characterisation of $\mathcal{C}$

Theorem. (Courcelle, Downey, Fellows 1997)

The following problem is undecidable:

Given: MSO-formula φ defining a minor ideal $\mathcal{C}$
Problem: Compute an excluded minor characterisation of $\mathcal{C}$

A positive answer. Excluded minor characterisations can be computed for a number of interesting classes.

A Negative Result

Theorem. (Fellows, Langston 1989)

The following problem is undecidable:

Given:	Turing machine M deciding a minor ideal $\mathcal{C}$
Problem:	Compute an excluded minor characterisation of $\mathcal{C}$

Theorem. (Courcelle, Downey, Fellows 1997)

The following problem is undecidable:

Given:	MSO-formula φ defining a minor ideal $\mathcal{C}$
Problem:	Compute an excluded minor characterisation of $\mathcal{C}$

A positive answer. Excluded minor characterisations can be computed for a number of interesting classes.

A Negative Result

Theorem. (Fellows, Langston 1989)

The following problem is undecidable:

Given: Turing machine M deciding a minor ideal $\mathcal{C}$
Problem: Compute an excluded minor characterisation of $\mathcal{C}$

Theorem. (Courcelle, Downey, Fellows 1997)

The following problem is undecidable:

Given: MSO-formula φ defining a minor ideal $\mathcal{C}$
Problem: Compute an excluded minor characterisation of $\mathcal{C}$

A positive answer. Excluded minor characterisations can be computed for a number of interesting classes.

Results

Here. General method based on definability in MSO for computing obstruction sets.

Applications. Given $k \geq 1$, the class of obstructions is computable for

- the class $\mathcal{T}_k$ of graphs of tree-width at most k

  (Lagergren '98)

- the class $\mathcal{B}_k$ of graphs of branch-width at most k

  (Geelen, Gerards, Robertson, Whittle '03)

- the class $\mathcal{G}_k$ of graphs of genus at most k

  (Seymour '95)

Constructions.

- Given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$ one can compute $\mathcal{O}(\mathcal{C} \cup \mathcal{D}), \mathcal{O}(\mathcal{C} \cap \mathcal{D})$
- Given $\mathcal{O}(\mathcal{C})$ one can compute $\mathcal{O}(\mathcal{C}^{k\text{-apex}})$ of graphs of distance $\leq k$ from $\mathcal{C}$

  Recall. G has distance k from $\mathcal{C}$: there are vertices $v_1, \dots, v_k \in V(G)$ such that $G - \{v_1, \dots, v_k\} \in \mathcal{C}$.

Results

Here. General method based on definability in MSO for computing obstruction sets.

Applications. Given $k \geq 1$, the class of obstructions is computable for

- the class $\mathcal{T}_k$ of graphs of tree-width at most k

  (Lagergren '98)

- the class $\mathcal{B}_k$ of graphs of branch-width at most k

  (Geelen, Gerards, Robertson, Whittle '03)

- the class $\mathcal{G}_k$ of graphs of genus at most k

  (Seymour '95)

Constructions.

- Given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$ one can compute $\mathcal{O}(\mathcal{C} \cup \mathcal{D}), \mathcal{O}(\mathcal{C} \cap \mathcal{D})$
- Given $\mathcal{O}(\mathcal{C})$ one can compute $\mathcal{O}(\mathcal{C}^{k\text{-apex}})$ of graphs of distance $\leq k$ from $\mathcal{C}$

  Recall. G has distance k from $\mathcal{C}$: there are vertices $v_1, \dots, v_k \in V(G)$ such that $G - \{v_1, \dots, v_k\} \in \mathcal{C}$.

Results

Here. General method based on definability in MSO for computing obstruction sets.

Applications. Given $k \geq 1$, the class of obstructions is computable for

- the class $\mathcal{T}_k$ of graphs of tree-width at most k (Lagergren '98)
- the class $\mathcal{B}_k$ of graphs of branch-width at most k (Geelen, Gerards, Robertson, Whittle '03)
- the class $\mathcal{G}_k$ of graphs of genus at most k (Seymour '95)

Constructions.

- Given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$ one can compute $\mathcal{O}(\mathcal{C} \cup \mathcal{D}), \mathcal{O}(\mathcal{C} \cap \mathcal{D})$
- Given $\mathcal{O}(\mathcal{C})$ one can compute $\mathcal{O}(\mathcal{C}^{k\text{-apex}})$ of graphs of distance $\leq k$ from $\mathcal{C}$

  Recall. G has distance k from $\mathcal{C}$: there are vertices $v_1, \dots, v_k \in V(G)$ such that $G - \{v_1, \dots, v_k\} \in \mathcal{C}$.

Results

Here. General method based on definability in MSO for computing obstruction sets.

Applications. Given $k \geq 1$, the class of obstructions is computable for

- the class $\mathcal{T}_k$ of graphs of tree-width at most k (Lagergren '98)
- the class $\mathcal{B}_k$ of graphs of branch-width at most k (Geelen, Gerards, Robertson, Whittle '03)
- the class $\mathcal{G}_k$ of graphs of genus at most k (Seymour '95)

Constructions.

- Given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$ one can compute $\mathcal{O}(\mathcal{C} \cup \mathcal{D}), \mathcal{O}(\mathcal{C} \cap \mathcal{D})$
- Given $\mathcal{O}(\mathcal{C})$ one can compute $\mathcal{O}(\mathcal{C}^{k\text{-apex}})$ of graphs of distance $\leq k$ from $\mathcal{C}$

  Recall. G has distance k from $\mathcal{C}$: there are vertices $v_1, \ldots, v_k \in V(G)$ such that $G - \{v_1, \ldots, v_k\} \in \mathcal{C}$.

Monadic Second-Order Logic

Monadic Second-Order Logic. Extension of first-order logic by set quantification.

Example. 3-COLOURABILITY

$$\underbrace{\exists C_1 \exists C_2 \exists C_3}_{\substack{\text{there are sets}\\ C_1, C_2, C_3}} \Big(\underbrace{\forall x \bigvee_{i=1}^{3} C_i(x)}_{\text{ev. node has a col.}} \quad \wedge \quad \underbrace{\forall x \forall y \big(E(x,y) \rightarrow \bigwedge_{i=1}^{3} \neg(C_i(x) \wedge C_i(y))\big)}_{\text{endpoints of edges have different colours}} \Big)$$

Example.

For every H there is an MSO-formula φ_H such that for all G

$$G \models \varphi_H \quad \text{iff} \quad H \preceq G$$

Lemma. If $\mathcal{C}$ is an MSO-definable minor ideal, then $\mathcal{O}(\mathcal{C})$ is MSO-definable also.

(Say: $G \notin \mathcal{C}$ but every proper (induced) subgraph and contraction is)

Monadic Second-Order Logic

Monadic Second-Order Logic. Extension of first-order logic by set quantification.

Example. 3-COLOURABILITY

$$\underbrace{\exists C_1 \exists C_2 \exists C_3}_{\substack{\text{there are sets} \\ C_1, C_2, C_3}} \Big(\underbrace{\forall x \bigvee_{i=1}^{3} C_i(x)}_{\text{ev. node has a col.}} \wedge \underbrace{\forall x \forall y \big(E(x,y) \rightarrow \bigwedge_{i=1}^{3} \neg(C_i(x) \wedge C_i(y))\big)}_{\text{endpoints of edges have different colours}} \Big)$$

Example.

For every H there is an MSO-formula φ_H such that for all G

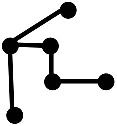
$$G \models \varphi_H \quad \text{iff} \quad H \preceq G$$

Lemma. If $\mathcal{C}$ is an MSO-definable minor ideal, then $\mathcal{O}(\mathcal{C})$ is MSO-definable also.

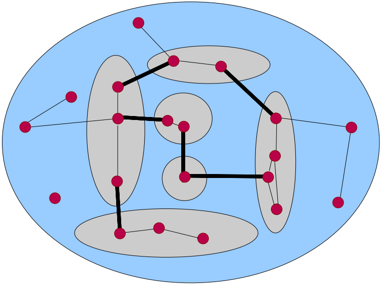
(Say: $G \notin \mathcal{C}$ but every proper (induced) subgraph and contraction is)

Minors

H:



G:



Minor Map: Function $\rho : V(H) \cup E(H) \longrightarrow \{G' : G' \subseteq G\}$ such that

- for each $v \in V(H)$,
  - $\rho(v) \subseteq G$ is a connected subgraphso that
  - $u \neq v$ implies $\rho(u) \cap \rho(v) = \emptyset$
- for each $e = \{u, v\} \in E(H)$,

  $\rho(e) := \{u', v'\} \in E(G)$ so that $u' \in \rho(u)$ and $v' \in \rho(v)$.

Monadic Second-Order Logic

Monadic Second-Order Logic. Extension of first-order logic by set quantification.

Example. 3-COLOURABILITY

$$\underbrace{\exists C_1 \exists C_2 \exists C_3}_{\substack{\text{there are sets} \\ C_1, C_2, C_3}} \Big(\underbrace{\forall x \bigvee_{i=1}^{3} C_i(x)}_{\text{ev. node has a col.}} \wedge \underbrace{\forall x \forall y \big(E(x,y) \rightarrow \bigwedge_{i=1}^{3} \neg(C_i(x) \wedge C_i(y))\big)}_{\text{endpoints of edges have different colours}} \Big)$$

Example.

For every H there is an MSO-formula φ_H such that for all G

$$G \models \varphi_H \quad \text{iff} \quad H \preceq G$$

Lemma. If $\mathcal{C}$ is an MSO-definable minor ideal, then $\mathcal{O}(\mathcal{C})$ is MSO-definable also.

(Say: $G \notin \mathcal{C}$ but every proper (induced) subgraph and contraction is)

Monadic Second-Order Logic

Monadic Second-Order Logic. Extension of first-order logic by set quantification.

Example. 3-COLOURABILITY

$$\underbrace{\exists C_1 \exists C_2 \exists C_3}_{\substack{\text{there are sets} \\ C_1, C_2, C_3}} \Big(\underbrace{\forall x \bigvee_{i=1}^{3} C_i(x)}_{\text{ev. node has a col.}} \wedge \underbrace{\forall x \forall y \big(E(x,y) \rightarrow \bigwedge_{i=1}^{3} \neg (C_i(x) \wedge C_i(y)) \big)}_{\text{endpoints of edges have different colours}} \Big)$$

Example.

For every H there is an MSO-formula φ_H such that for all G

$$G \models \varphi_H \quad \text{iff} \quad H \preceq G$$

Lemma. If $\mathcal{C}$ is an MSO-definable minor ideal, then $\mathcal{O}(\mathcal{C})$ is MSO-definable also.

(Say: $G \notin \mathcal{C}$ but every proper (induced) subgraph and contraction is)

Defining Obstructions

Lemma. If $\mathcal{C}$ is an MSO-definable minor ideal, then $\mathcal{O}(\mathcal{C})$ is MSO-definable also.

(Say: $G \notin \mathcal{C}$ but every proper (induced) subgraph and contraction is)

Lemma. If $\varphi \in \mathrm{MSO}$ has only finitely many models and k is an upper bound of the tree-width of its models then an upper bound for the size of its models can be computed from φ, k.

Furthermore, there is an algorithm that computes all models of φ.

Consequence. If $\mathcal{C}$ is a minor ideal such that

- $\mathcal{C}$ is definable by $\varphi \in \mathrm{MSO}$ and
- the maximal tree-width of any of its obstructions is k

then the obstructions for $\mathcal{C}$ can be computed from k and φ.

Defining Obstructions

Lemma. If $\mathcal{C}$ is an MSO-definable minor ideal, then $\mathcal{O}(\mathcal{C})$ is MSO-definable also.

(Say: $G \notin \mathcal{C}$ but every proper (induced) subgraph and contraction is)

Lemma. If $\varphi \in \text{MSO}$ has only finitely many models and k is an upper bound of the tree-width of its models then an upper bound for the size of its models can be computed from φ, k.

Furthermore, there is an algorithm that computes all models of φ.

Consequence. If $\mathcal{C}$ is a minor ideal such that

- $\mathcal{C}$ is definable by $\varphi \in \text{MSO}$ and
- the maximal tree-width of any of its obstructions is k

then the obstructions for $\mathcal{C}$ can be computed from k and φ.

Obstructions of Definable Classes

Definition. Let $\mathcal{C}$ be a minor ideal.

The obstruction width or O-width of $\mathcal{C}$ is the maximal tree-width of any of its obstructions.

Main lemma. Let $(\mathcal{C}_i)_{i \in I}$ be a family of minor ideals such that

1. the O-width of $\mathcal{C}_i$ is $\leq f(i)$ for some computable function f
2. $\mathcal{C}_i$ is definable by MSO-formula φ_i, for a comp. function $i \mapsto \varphi_i$

Then there is an algorithm which, given $i \in I$ computes $\mathcal{O}(\mathcal{C}_i)$.

Example. Let $I := \mathbb{N}$ and $\mathcal{C}_i := \mathcal{T}_i$, the graphs of tree-width $\leq i$.

1. The O-width of $\mathcal{C}_i$ is $i + 1$.
2. $\mathcal{C}_i$ is MSO-definable (within suitable expansions)

Remark. A similar result in terms of Myhill-Nerode congruences has been established by Fellows and Langston 1989.

Obstructions of Definable Classes

Definition. Let $\mathcal{C}$ be a minor ideal.

The obstruction width or O-width of $\mathcal{C}$ is the maximal tree-width of any of its obstructions.

Main lemma. Let $(\mathcal{C}_i)_{i \in I}$ be a family of minor ideals such that

1. the O-width of $\mathcal{C}_i$ is $\leq f(i)$ for some computable function f
2. $\mathcal{C}_i$ is definable by MSO-formula φ_i, for a comp. function $i \mapsto \varphi_i$

Then there is an algorithm which, given $i \in I$ computes $\mathcal{O}(\mathcal{C}_i)$.

Example. Let $I := \mathbb{N}$ and $\mathcal{C}_i := \mathcal{T}_i$, the graphs of tree-width $\leq i$.

1. The O-width of $\mathcal{C}_i$ is $i + 1$.
2. $\mathcal{C}_i$ is MSO-definable (within suitable expansions)

Remark. A similar result in terms of Myhill-Nerode congruences has been established by Fellows and Langston 1989.

Obstructions of Definable Classes

Definition. Let $\mathcal{C}$ be a minor ideal.

The obstruction width or O-width of $\mathcal{C}$ is the maximal tree-width of any of its obstructions.

Main lemma. Let $(\mathcal{C}_i)_{i \in I}$ be a family of minor ideals such that

1. the O-width of $\mathcal{C}_i$ is $\leq f(i)$ for some computable function f
2. $\mathcal{C}_i$ is definable by MSO-formula φ_i, for a comp. function $i \mapsto \varphi_i$

Then there is an algorithm which, given $i \in I$ computes $\mathcal{O}(\mathcal{C}_i)$.

Example. Let $I := \mathbb{N}$ and $\mathcal{C}_i := \mathcal{T}_i$, the graphs of tree-width $\leq i$.

1. The O-width of $\mathcal{C}_i$ is $i + 1$.
2. $\mathcal{C}_i$ is MSO-definable (within suitable expansions)

Remark. A similar result in terms of Myhill-Nerode congruences has been established by Fellows and Langston 1989.

Results

Here. General method based on definability in MSO for computing obstruction sets.

Applications. Given $k \geq 1$, the class of obstructions is computable for

- the class $\mathcal{T}_k$ of graphs of tree-width at most k
  (Lagergren '98)
- the class $\mathcal{B}_k$ of graphs of branch-width at most k
  (Geelen, Gerards, Robertson, Whittle '03)
- the class $\mathcal{G}_k$ of graphs of genus at most k
  (Seymour '95)

Constructions.

- Given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$ one can compute $\mathcal{O}(\mathcal{C} \cup \mathcal{D}), \mathcal{O}(\mathcal{C} \cap \mathcal{D})$
- Given $\mathcal{O}(\mathcal{C})$ one can compute $\mathcal{O}(\mathcal{C}^{k\text{-apex}})$ of graphs of distance $\leq k$ from $\mathcal{C}$

  Recall. G has distance k from $\mathcal{C}$: there are vertices $v_1, \dots, v_k \in V(G)$ such that $G - \{v_1, \dots, v_k\} \in \mathcal{C}$.

Union and Intersection of Minor Ideals

Let $\mathcal{C}, \mathcal{D}$ be minor ideals.

Intersection. $\mathcal{C} \cap \mathcal{D}$ is a minor ideal characterised by $\mathcal{O}(\mathcal{C}) \cup \mathcal{O}(\mathcal{D})$.

$\mathcal{O}\big(\mathcal{C} \cap \mathcal{D}\big)$ can easily be computed from $\mathcal{O}(\mathcal{C}) \cup \mathcal{O}(\mathcal{D})$.

Union. $\mathcal{C} \cup \mathcal{D}$ is a minor ideal.

Question. (Fellows, Langston 1989)

Can $\mathcal{O}(\mathcal{C} \cup \mathcal{D})$ be computed from $\mathcal{O}(\mathcal{C})$ and $\mathcal{O}(\mathcal{D})$?

Union and Intersection of Minor Ideals

Let $\mathcal{C}, \mathcal{D}$ be minor ideals.

Intersection. $\mathcal{C} \cap \mathcal{D}$ is a minor ideal characterised by $\mathcal{O}(\mathcal{C}) \cup \mathcal{O}(\mathcal{D})$.

$\mathcal{O}\big(\mathcal{C} \cap \mathcal{D}\big)$ can easily be computed from $\mathcal{O}(\mathcal{C}) \cup \mathcal{O}(\mathcal{D})$.

Union. $\mathcal{C} \cup \mathcal{D}$ is a minor ideal.

Question. (Fellows, Langston 1989)

Can $\mathcal{O}(\mathcal{C} \cup \mathcal{D})$ be computed from $\mathcal{O}(\mathcal{C})$ and $\mathcal{O}(\mathcal{D})$?

Minor Characterisation for Unions of Ideals

Let $\mathcal{C}, \mathcal{D}$ be minor ideals.

Lemma. There is a computable function f such that

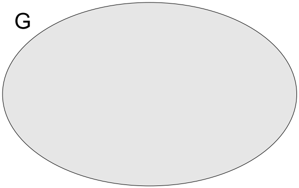
$$\text{O-width}(\mathcal{C} \cup \mathcal{D}) \leq f(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D}))$$

Proof. Show: if $G \notin \mathcal{C} \cup \mathcal{D}$ then G contains a subgraph $G' \notin \mathcal{C} \cup \mathcal{D}$ of tree-width $\leq f(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D}))$.

There are $H' \in \mathcal{O}(\mathcal{C})$ and $I' \in \mathcal{O}(\mathcal{D})$ such that $H \preceq G$ and $I \preceq G$.

Let H, I be models of H', I' in G and $G' := H \cup I \subseteq G$.

Show that G' has small tree-width.



Minor Characterisation for Unions of Ideals

Let $\mathcal{C}, \mathcal{D}$ be minor ideals.

Lemma. There is a computable function f such that

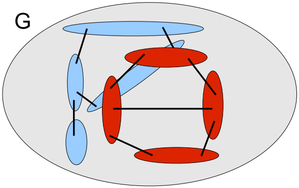
$$\text{O-width}\big(\mathcal{C} \cup \mathcal{D}\big) \leq f\big(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})\big)$$

Proof. Show: if $G \notin \mathcal{C} \cup \mathcal{D}$ then G contains a subgraph $G' \notin \mathcal{C} \cup \mathcal{D}$ of tree-width $\leq f\big(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})\big)$.

There are $H' \in \mathcal{O}(\mathcal{C})$ and $I' \in \mathcal{O}(\mathcal{D})$ such that $H \preceq G$ and $I \preceq G$.

Let H, I be models of H', I' in G and $G' := H \cup I \subseteq G$.

Show that G' has small tree-width.



Minor Characterisation for Unions of Ideals

Let $\mathcal{C}, \mathcal{D}$ be minor ideals.

Lemma. There is a computable function f such that

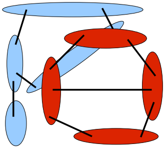
$$\text{O-width}\big(\mathcal{C} \cup \mathcal{D}\big) \leq f\big(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})\big)$$

Proof. Show: if $G \notin \mathcal{C} \cup \mathcal{D}$ then G contains a subgraph $G' \notin \mathcal{C} \cup \mathcal{D}$ of tree-width $\leq f\big(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})\big)$.

There are $H' \in \mathcal{O}(\mathcal{C})$ and $I' \in \mathcal{O}(\mathcal{D})$ such that $H \preceq G$ and $I \preceq G$.

Let H, I be models of H', I' in G and $G' := H \cup I \subseteq G$.

Show that G' has small tree-width.



The Trinity Lemma

Trinity Lemma. (Robertson and Seymour)

There is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all graphs G and all k

1. G has tree-width $\leq f(k)$ or
2. $K_k \preceq G$ or
3. G contains a flat $k \times k$-grid as a minor

Application. Apply the lemma to G' with $k := \max\{|H'|, |I'|\}$

1. G' has tree-width $\leq f(k)$
2. Easily seen that G' contains subgraph of tree-width $\leq k^2 + 1$ containing a K_k minor.
3. Another lemma by Robertson and Seymour implies that we can remove a vertex $v \in V(G)'$ so that $G' - v \notin \mathcal{C} \cup \mathcal{D}$.

The Trinity Lemma

Trinity Lemma. (Robertson and Seymour)

There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all graphs G and all k

1. G has tree-width $\leq f(k)$ or
2. $K_k \preceq G$ or
3. G contains a flat $k \times k$-grid as a minor

Application. Apply the lemma to G' with $k := \max\{|H'|, |I'|\}$

1. G' has tree-width $\leq f(k)$
2. Easily seen that G' contains subgraph of tree-width $\leq k^2 + 1$ containing a K_k minor.
3. Another lemma by Robertson and Seymour implies that we can remove a vertex $v \in V(G)'$ so that $G' - v \notin \mathcal{C} \cup \mathcal{D}$.

Minor Characterisation for Unions of Ideals

We aim at applying the main lemma to compute $\mathcal{O}(\mathcal{C} \cup \mathcal{D})$

Main lemma. Let $(\mathcal{C}_i)_{i \in I}$ be a family of minor ideals such that

1. the O-width of $\mathcal{C}_i$ is $\leq f(i)$ for some computable function f
2. $\mathcal{C}_i$ is definable by MSO-formula φ_i, for a comp. function $i \mapsto \varphi_i$

Then there is an algorithm which, given $i \in I$ computes $\mathcal{O}(\mathcal{C}_i)$.

Set $\mathcal{I} := \{(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})) : \mathcal{C}, \mathcal{D}$ minor ideals $\}$

$\mathcal{C}_i := \mathcal{C} \cup \mathcal{D}$ where $i := (\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D}))$

Previous lemma. There is a computable function f such that

$$\text{O-width}(\mathcal{C} \cup \mathcal{D}) \leq f(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D}))$$

MSO-definability. $\mathcal{C}_i$ is easily MSO-definable given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$.

Consequence. Given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$ we can compute $\mathcal{O}(\mathcal{C} \cup \mathcal{D})$

Minor Characterisation for Unions of Ideals

We aim at applying the main lemma to compute $\mathcal{O}\big(\mathcal{C} \cup \mathcal{D}\big)$

Main lemma. Let $(\mathcal{C}_i)_{i \in I}$ be a family of minor ideals such that

1. the O-width of $\mathcal{C}_i$ is $\leq f(i)$ for some computable function f
2. $\mathcal{C}_i$ is definable by MSO-formula φ_i, for a comp. function $i \mapsto \varphi_i$

Then there is an algorithm which, given $i \in I$ computes $\mathcal{O}(\mathcal{C}_i)$.

Set $\mathcal{I} := \{\big(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})\big) : \mathcal{C}, \mathcal{D}$ minor ideals $\}$

$\mathcal{C}_i := \mathcal{C} \cup \mathcal{D}$ where $i := \big(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})\big)$

Previous lemma. There is a computable function f such that

$$\text{O-width}\big(\mathcal{C} \cup \mathcal{D}\big) \leq f\big(\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})\big)$$

MSO-definability. $\mathcal{C}_i$ is easily MSO-definable given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$.

Consequence. Given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$ we can compute $\mathcal{O}\big(\mathcal{C} \cup \mathcal{D}\big)$

Results

Here. General method based on definability in MSO for computing obstruction sets.

Applications. Given $k \geq 1$, the class of obstructions is computable for

- the class $\mathcal{T}_k$ of graphs of tree-width at most k
  (Lagergren '98)
- the class $\mathcal{B}_k$ of graphs of branch-width at most k
  (Geelen, Gerards, Robertson, Whittle '03)
- the class $\mathcal{G}_k$ of graphs of genus at most k
  (Seymour '95)

Constructions.

- Given $\mathcal{O}(\mathcal{C}), \mathcal{O}(\mathcal{D})$ one can compute $\mathcal{O}(\mathcal{C} \cup \mathcal{D}), \mathcal{O}(\mathcal{C} \cap \mathcal{D})$
- Given $\mathcal{O}(\mathcal{C})$ one can compute $\mathcal{O}(\mathcal{C}^{k\text{-apex}})$ of graphs of distance $\leq k$ from $\mathcal{C}$

  Recall. G has distance k from $\mathcal{C}$: there are vertices $v_1, \dots, v_k \in V(G)$ such that $G - \{v_1, \dots, v_k\} \in \mathcal{C}$.

Open Problems

General Open Problem.

Show that obstruction sets can be computed for other natural minor ideals.

Specific Problem.

- Excluded minors for the classes $\mathcal{L}(\mu)$ of linearly bounded tree-width.
- Given minor ideal $\mathcal{C}$, compute the obstructions for the class of graphs having a tree-decomposition over $\mathcal{C}$

Practise. (Here I speak for myself, I guess ...)

Compute excluded minors for tree-width 4 and more...

The results may not be applicable as it is but there is hope to make it practical.

Open Problems

General Open Problem.

Show that obstruction sets can be computed for other natural minor ideals.

Specific Problem.

- Excluded minors for the classes $\mathcal{L}(\mu)$ of linearly bounded tree-width.
- Given minor ideal $\mathcal{C}$, compute the obstructions for the class of graphs having a tree-decomposition over $\mathcal{C}$

Practise. (Here I speak for myself, I guess ...)

Compute excluded minors for tree-width 4 and more...

The results may not be applicable as it is but there is hope to make it practical.

Open Problems

General Open Problem.

Show that obstruction sets can be computed for other natural minor ideals.

Specific Problem.

- Excluded minors for the classes $\mathcal{L}(\mu)$ of linearly bounded tree-width.
- Given minor ideal $\mathcal{C}$, compute the obstructions for the class of graphs having a tree-decomposition over $\mathcal{C}$

Practise. (Here I speak for myself, I guess ...)

Compute excluded minors for tree-width 4 and more...

The results may not be applicable as it is but there is hope to make it practical.