Hard CSPs have hard gaps at location 1

Andrei Krokhin
Durham University, UK

Joint work with
Peter Jonsson and Fredrik Kuivinen
Linköping University, Sweden

Constraints

- D – a finite set with $|D| > 1$;
- $R_D^{(m)}$ = subsets of D^m, $R_D = \bigcup_{m=1}^{\infty} R_D^{(m)}$.

Definition 1 *A constraint over a set of variables* $V = \{x_1, x_2, \ldots, x_n\}$ *is a pair of the form* $C = (\mathbf{x}, \varrho)$ *where*

- $\mathbf{x} = (x_{i_1}, \ldots, x_{i_m})$ *is the constraint scope,*
- $\varrho \in R_D^{(m)}$ *is the constraint relation.*

The constraint C *is said to be satisfied by an assignment* $f : V \to D$ *if* $(f(x_{i_1}), \ldots, f(x_{i_m})) \in \varrho$.

The Constraint Satisfaction Problem

CSP

Instance: A collection $C_1, \ldots, C_q$ of constraints over V.

Question: Is there an assignment $f : V \to D$ satisfying all these constraints?

MAX CSP

Instance: A collection $C_1, \ldots, C_q$ of constraints over V.

Goal: Find an assignment $f : V \to D$ satisfying maximum number of the constraints?

Parameterisation of CSP and MAX CSP

Definition 2 *A constraint language is finite subset of* R_D*.*

For a constraint language Γ, CSP(Γ) *and* Max CSP(Γ) *consist of all* CSP *and* MAX CSP, *respectively, instances in which all constraint relations belong to* Γ*.*

Research Programme

Classify the complexity and approximability of the problems CSP(Γ) and Max CSP(Γ).

Disclaimer: We assume $\mathbf{P} \neq \mathbf{NP}$ throughout.

The bounded occurrence property

Definition 3 *Let* CSP(Γ)-k *(*Max CSP(Γ)-k*) denote the problem* CSP(Γ) *(*Max CSP(Γ)*, respectively) restricted to instances where the number of occurrences of each variable (counted with multiplicity of constraints) is bounded by* k*.*

NB. This is very similar to restricting graph problems to classes of graphs of bounded degree.

Definition 4 *We say that* CSP(Γ)-B *(*Max CSP(Γ)-B*) is hard (in some sense) if there exists a number* k *such that* CSP(Γ)-k *(*Max CSP(Γ)-k*, resp.) is hard in that sense.*

Example: 2-COL and MAX CUT

Let $D = \{0,1\}$ and $\Gamma = \{neq\}$ where $(x,y) \in neq$ iff $x \neq y$.

Then CSP(Γ) is the 2-COLOURABILITY problem and Max CSP(Γ) is precisely the MAX CUT problem.

For an instance $\mathcal{I}$ of CSP(Γ) over $V = \{x_1, \ldots, x_n\}$, consider a (multi)graph $G_{\mathcal{I}} = (V, E)$ with E consisting of constraint scopes in $\mathcal{I}$.

Clearly, $\mathcal{I}$ is satisfiable iff $G_{\mathcal{I}}$ is 2-colourable.
Moreover, computing maximum cut in $G_{\mathcal{I}}$ is the same as maximising the number of satisfied constraints in $\mathcal{I}$.

Complexity: 2-COL is in **P**, MAX CUT-3 is **NP**-hard.

Example: 3-Sat and Max 3-Sat

Let $D = \{0,1\}$ and let $\Gamma_{3sat} = \{\varrho_0, \varrho_1, \varrho_2, \varrho_3\}$ where

- $\varrho_0 = \{0,1\}^3 \setminus \{(0,0,0)\}$ $x \vee y \vee z$
- $\varrho_1 = \{0,1\}^3 \setminus \{(0,0,1)\}$ $x \vee y \vee \overline{z}$
- $\varrho_2 = \{0,1\}^3 \setminus \{(0,1,1)\}$ $x \vee \overline{y} \vee \overline{z}$
- $\varrho_3 = \{0,1\}^3 \setminus \{(1,1,1)\}$ $\overline{x} \vee \overline{y} \vee \overline{z}$

It is easy to see that $\mathrm{CSP}(\Gamma_{3sat})$ is precisely 3-Sat and Max $\mathrm{CSP}(\Gamma_{3sat})$ is precisely Max 3-Sat.

Complexity: Both problems are **NP**-hard.

The complexity classification problem

Conjecture 1 (Feder,Vardi '98) *Dichotomy conjecture: Each problem* CSP(Γ) *is either in* **P** *or else* **NP***-complete.*

Theorem 1 (Bulatov,Jeavons,K. '05) *If* Γ *has property* (G-SET) *then* CSP(Γ) *is* **NP***-complete.*

Conjecture 2 (BJK'05) *Algebraic dichotomy conjecture: If* Γ *does not have property* (G-SET) *then* CSP(Γ) *is in* **P**.

Theorem 2 (Bulatov '03-06) *Conjecture 2 holds when* $|D| \leq 3$ *or when* Γ *contains all unary relations.*

A property equivalent to $(G\text{-}\textsc{set})$

Assume wlog that Γ is a core, and let $\mathcal{C}_D = \{\{d\} \mid d \in D\}$. Recall the relations $\varrho_0, \varrho_1, \varrho_2, \varrho_3$ from Γ_{3sat}.

Then Γ has property $(G\text{-}\textsc{set})$ iff there exist

1. a subset U of D and a function $h : U \twoheadrightarrow \{0, 1\}$, and
2. four pp-formulas (=conjunctive queries) over $\Gamma \cup \mathcal{C}_D$ expressing precisely the relations

$$h^{-1}(\varrho_j) = \{(a, b, c) \in U^3 \mid (h(a), h(b), h(c)) \in \varrho_j\}.$$

A property opposite to (G-SET)

A weak near-unanimity (WNU) operation on D is an n-ary ($n \geq 2$) operation which satisfies the identities $f(x,\ldots,x) = x$ and $f(y,x,\ldots,x) = \ldots = f(x,\ldots,x,y)$.

Examples: $\min(x_1,\ldots,x_n)$, $\quad x_1 + \ldots + x_n + x_{n+1} (\mathrm{mod}\ n)$.

Recall that a polymorphism of Γ is an operation that preserves every relation in Γ.

Theorem 3 (Maróti,McKenzie '07)
A core Γ does not have property (G-SET) iff it has a WNU polymorphism of some arity.

The approximability classification problem

Fact 1 *For each problem* Max CSP(Γ)*, there exist a constant* $c_\Gamma \leq 1$ *and a poly-time* c_Γ*-approximation algorithm (i.e., producing a solution of value at least* $c_\Gamma \cdot OPT(\mathcal{I})$ *for every instance* $\mathcal{I}$ *of* Max CSP(Γ)*).*

Problem 1 *Characterise sets* Γ *such that*

- Max CSP(Γ) *is in* **PO** *(i.e.,* $c_\Gamma = 1$*)*
- Max CSP(Γ) *is* **NP***-hard and*
  - Max CSP(Γ) *has a PTAS – polynomial time approximation scheme (i.e.,* c_Γ *can be chosen arbitrarily close to 1)*
  - $c_\Gamma \leq \delta < 1$ – *"hard to approximate"*

Hard gap at location 1

Definition 5 *A problem* Max CSP(Γ) *is said to have a hard gap at location 1 if, for some fixed* $\alpha < 1$, *it is* **NP***-hard to distinguish between*

- *instances where all constraints can be satisfied, and*
- *those where at most* α*-fraction can be satisfied.*

Fact 2 *If* Max CSP(Γ) *has a hard gap at location 1 then*

- $c_\Gamma \leq \alpha < 1$ — *hard to approximate (even when restricted to satisfiable instances);*
- Max CSP(Γ) *cannot have a PTAS;*
- CSP(Γ) *cannot be in* **P**.

Relating to the PCP theorem

Theorem 4 (Arora et al' 98, Arora,Safra '98, Dinur'07) *The following equivalent statements hold:*

1. $\mathbf{NP} \subseteq \mathbf{PCP}[\log n, 1]$,
2. *for some constraint language* Γ *over some* D, Max CSP(Γ) *has a hard gap at location 1,*
3. MAX 3-SAT *has a hard gap at location 1.*

The proof of equivalence of the statements is quite easy (half a page), while the proof of validity is very hard.

Recent combinatorial proof of (2) by Dinur deals entirely with CSPs.

Main result

Theorem 5 *If* Γ *has property* (G-SET) *then the problem* Max CSP(Γ)-B *has a hard gap at location 1.*

Note that if the algebraic dichotomy conjecture holds then Max CSP(Γ) has a hard gap at location 1 for all Γ with hard CSP(Γ).

Corollary 1 *If* Γ *has property* (G-SET) *then the problem* Max CSP(Γ)-B *is hard to approximate even when it is restricted to satisfiable instances. In particular,* Max CSP(Γ)-B *has no PTAS.*

Key elements in proof

- Recall that property (G-SET) for a core Γ implies that $\Gamma \cup \mathcal{C}_D$ pp-expresses pre-images of relations from Γ_{3sat}.
- Hard gap for MAX 3-SAT$-B$ is the base case.
- Moving to pre-images for free
- Show that the presence of a hard gap is preserved when adding $\mathcal{C}_D$ and pp-expressed relations

Adding pp-expressed relations

Lemma 1 (Jeavons'98) *If a constraint language* Γ *pp-expresses a relation* ϱ *then* $\mathrm{CSP}(\Gamma \cup \{\varrho\})$ *poly-time reduces to* $\mathrm{CSP}(\Gamma)$.

The above also holds in the bounded occurrence setting.

Lemma 2 *If a constraint language* Γ *pp-expresses* ϱ *and* $\mathrm{Max\ CSP}(\Gamma \cup \{\varrho\})$*-*$k$ *has hard gap at location 1 then, for some* k', $\mathrm{Max\ CSP}(\Gamma)$*-*$k'$ *has hard gap at location 1.*

The gap parameter α becomes $\alpha' = \alpha + (1-\alpha)(1-1/N)$ where N is the number of relations in pp-expression for ϱ.

Adding $\mathcal{C}_D$

Lemma 3 (Bulatov,Jeavons,AK '05) *If* Γ *is a core then* $\mathrm{CSP}(\Gamma \cup \mathcal{C}_D)$ *poly-time reduces to* $\mathrm{CSP}(\Gamma)$.

The transformation in this lemma does not preserve the bounded occurrence property.
The proof (of the main theorem) gets around this.

All in all, the new gap parameter α' can be computed from

- the MAX 3-SAT gap parameter,
- the size of the domain $|D|$,
- a certain constant from expander graph construction,
- the number of relations in 5 pp-expressions from Γ.

One application

Theorem 6 *Let* $\varrho \in R_D$ *be non-empty and let* $\Gamma = \{\varrho\}$. *If* $(d, \ldots, d) \in \varrho$ *for some* $d \in D$ *then* Max CSP(Γ) *is trivial. Otherwise,* Max CSP$(\Gamma)$$-B$ *is hard to approximate.*

MAX CUT (= Max CSP$(\{neq\})$) is hard to approximate. Theorem 6 can be seen as a generalisation of this.

The proof is based on the main theorem, and uses the bounded occurrence property (in the main theorem) in an essential way.