Quantifying Chandra-Merlin

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Logic and Algorithms - July 2008

Chandra-Merlin: Preliminaries

- Throughout: $\mathbf{A}$, $\mathbf{B}$, … denote relational structures assumed to have *finite* universes $A, B, \ldots$
- Use CSP($\mathbf{B}$) to denote the set of all true *primitive positive* sentences ($\exists$, $\wedge$) over $\mathbf{B}$
- Use $Q_{\mathbf{A}}$ to denote the *canonical query* of the structure $\mathbf{A}$:

$$\exists a_1 \exists a_2 \ldots (R(a_{i_1}, \ldots, a_{i_k}) \wedge \ldots)$$

  where:

  - the elements $a_1, a_2, \ldots$ of $\mathbf{A}$ are all $\exists$ quantified
  - conjunction is over all symbols R and tuples $(a_{i_1}, \ldots, a_{i_k}) \in R^{\mathbf{A}}$

Chandra-Merlin

- Theorem (a version of Chandra-Merlin '77). Let $\mathbf{A}, \mathbf{B}$ be reln. structures. The following are equivalent:
  - $\mathrm{CSP}(\mathbf{A}) \subseteq \mathrm{CSP}(\mathbf{B})$ – *model containment*
  - $Q_{\mathbf{B}} \models Q_{\mathbf{A}}$ – *query containment*
  - $B \models Q_{\mathbf{A}}$, equivalently, $Q_{\mathbf{A}} \in \mathrm{CSP}(\mathbf{B})$ – *query evaluation*

  Problems all NP-complete in general.
- This talk: consider complexity of these logical problems, but where queries are *conjunctive positive*: formed with $\forall$, $\exists$, $\wedge$
- Natural problems - e.g. second is entailment in FO logic
- Recent progress on understanding complexity of all three
  - New decidability results on first, second

CSP and QCSP

- Use CSP($\mathbf{B}$) to denote the set of all true *primitive positive* sentences ($\exists$, $\wedge$) over $\mathbf{B}$
- Use QCSP($\mathbf{B}$) to denote the set of all true *conjunctive positive* sentences ($\forall$, $\exists$, $\wedge$) over $\mathbf{B}$

Model Containment: CSP and QCSP

- Theorem (from Chandra-Merlin): The following are equivalent.
  1. $\mathrm{CSP}(\mathbf{A}) \subseteq \mathrm{CSP}(\mathbf{B})$
  2. $\mathbf{A} \to \mathbf{B}$
- Theorem (ChenMadelaineMartin '08) \{ Chen }: The following are equivalent.
  1. $\mathrm{QCSP}(\mathbf{A}) \subseteq \mathrm{QCSP}(\mathbf{B})$
  2. $\exists r(\mathbf{A}^r \stackrel{\text{surj}}{\to} \mathbf{B})$
- Proof of (2) $\Rightarrow$ (1) "easier" direction. Ingredients:
  - Lemma: If $\mathbf{A} \stackrel{\text{surj}}{\to} \mathbf{B}$ then $\mathrm{QCSP}(\mathbf{A}) \subseteq \mathrm{QCSP}(\mathbf{B})$ (proved with strategy simulation)
  - Lemma: $(\forall r \geq 1)\ \mathrm{QCSP}(\mathbf{A}) = \mathrm{QCSP}(\mathbf{A}^r)$

Model Containment: Proof

To prove: $\text{QCSP}(\mathbf{A}) \subseteq \text{QCSP}(\mathbf{B}) \Rightarrow \exists r(\mathbf{A}^r \stackrel{\text{surj}}{\rightarrow} \mathbf{B})$

- Let $B = \{b_1, \ldots, b_{|B|}\}$
- Let $k = |A|^{|B|}$. Let $\lambda_1, \ldots, \lambda_k$ be all maps $B \to A$.
- Consider formula similar to canon. query of $\mathbf{A}^k$:
$$\Phi \equiv \forall y_1 \ldots \forall y_{|B|} \exists \ldots (\mathbf{A}^k \to \mathbf{C})$$
where $y_i = (\lambda_1(b_i), \ldots, \lambda_k(b_i))$
- Φ true in $\mathbf{A}$:
  - No matter how y_i are set, *some* λ_j sends $b_i \to y_i$
  - Set $\exists$ variables according to jth projection
  - Then all vars. are set to jth projection which is hom. $\mathbf{A}^k \to \mathbf{A}$
- Suppose Φ true in $\mathbf{B}$.
  - Consider the assignment $y_i \to b_i$
  - This has an extension that is a hom. $\mathbf{A}^k \to \mathbf{B}$
  - And this extension is surjective!

Model Containment: Obtained Theorem

Theorem (ChenMadelaineMartin '08): The following are equivalent.

1. QCSP($\mathbf{A}$) $\subseteq$ QCSP($\mathbf{B}$)
2. $\exists r (\mathbf{A}^r \stackrel{\text{surj}}{\to} \mathbf{B})$
3. $\mathbf{A}^{|A|^{|B|}} \stackrel{\text{surj}}{\to} \mathbf{B}$
   (decidable condition!)
4. Π_2-QCSP($\mathbf{A}$) $\subseteq$ Π_2-QCSP($\mathbf{B}$)

Query Containment

Let ϕ, ψ be sentences (of some form).
What is the complexity of deciding if ϕ entails ψ: $\phi \models \psi$?

- Primitive positive ($\exists$, $\wedge$) sentences: NP-complete
- Conjunctive positive ($\forall$, $\exists$, $\wedge$) sentences: decidable
  - New decidability result for entailment in FO logic! (ChenMadelaineMartin '08)
  - Also get decidability for ϕ conjunctive positive, ψ positive
- Positive ($\forall$, $\exists$, $\wedge$, $\vee$) sentences: undecidable
  - Proof in (ChenMadelaineMartin '08)

Canonical Model

- Want to know if $\phi \models \psi$ for conjunctive positive sentences
- Idea: build a "canonical model" T_ϕ for ϕ then check if $T_\phi \models \psi$
- Let $\phi = \forall y_1 \exists x_1 \ldots \forall y_k \exists x_k (R(y_1, x_1, y_2, x_2, \ldots) \wedge \ldots)$
- Define $T_\phi(C)$ to be the set of all terms over constants C and function symbols $\{f_1, \ldots, f_k\}$ of arity $1, \ldots, k$
- Then create a structure: for the atom $R(y_1, x_1, y_2, x_2, \ldots)$ we add to R^{T_ϕ} all tuples

$$(y_1, f_1(y_1), y_2, f_2(y_1, y_2), \ldots)$$

over all $y_1, y_2, \ldots \in T_\phi$

Why is this a model of ϕ?

- Why does it hold that $T_\phi(C) \models \phi$?
- View $\phi = \forall y_1 \exists x_1 \ldots \forall y_k \exists x_k (R(y_1, x_1, y_2, x_2, \ldots) \wedge \ldots)$ as a $\forall$, $\exists$ game:
  - $\forall$ sets y_1, $\exists$ sets x_1, ...
  - $\exists$ tries to satisfy quant-free part $(R(\cdot) \wedge \ldots)$
- Whatever values $b_1, b_2, \ldots$ are played by $\forall$, $\exists$ can play $f_1(b_1), f_2(b_1, b_2), \ldots$

Towards decidability

- Define sets of constants:
  $C_\omega = \{c_1, c_2, \ldots\}$
  $C_m = \{c_1, c_2, \ldots, c_m\}$
- Theorem: Let ϕ, ψ be conjunctive positive, let ψ have m universal variables. The following are equivalent:
  1. $\phi \models \psi$
  2. $\exists$ wins ψ-game on $T_\phi(C_\omega)$ where $\forall$ plays constants
  3. $\exists$ wins ψ-game on $T_\phi(C_\omega)$ where $\forall$ plays constants, $\exists$ is "constant conservative" (only uses constants already in play)
  4. $\exists$ wins ψ-game on $T_\phi(C_m)$ where $\forall$ plays constants, $\exists$ is "constant conservative"

Proof

- Theorem: Let ϕ, ψ be conjunctive positive, let ψ have m universal variables. The following are equivalent:
  1. $\phi \models \psi$
  2. $\exists$ wins ψ-game on $T_\phi(C_\omega)$ where $\forall$ plays constants
  3. $\exists$ wins ψ-game on $T_\phi(C_\omega)$ where $\forall$ plays constants, $\exists$ is "constant conservative"
  4. $\exists$ wins ψ-game on $T_\phi(C_m)$ where $\forall$ plays constants, $\exists$ is "constant conservative"
- Proof (1) $\Rightarrow$ (2):
  - Have fact that $T_\phi(C_\omega) \models \phi$
  - Then have $T_\phi(C_\omega) \models \psi$,
    i.e. $\exists$ wins ψ-game on $T_\phi(C_\omega)$

Proof

- Theorem: Let ϕ, ψ be conjunctive positive, let ψ have m universal variables. The following are equivalent:
  1. $\phi \models \psi$
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  3. $\exists$ wins ψ-game on $T_\phi(C_\omega)$ where $\forall$ plays constants, $\exists$ is "constant conservative"
  4. $\exists$ wins ψ-game on $T_\phi(C_m)$ where $\forall$ plays constants, $\exists$ is "constant conservative"
- Proof (2) $\Rightarrow$ (3). Idea: Take an $\exists$ winning strategy in the form of game tree (play starts from top), and perform surgery.
  - At a point where $\exists$ plays a "new" constant c, remove all "lower" universal plays of c
  - Then, everywhere below the point, perform a "constant substitution": substitute c for a constant already played
  - Replace "missing" subtrees by using symmetry of constants

Proof

- Theorem: Let ϕ, ψ be conjunctive positive, let ψ have m universal variables. The following are equivalent:
  1. $\phi \models \psi$
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  3. $\exists$ wins ψ-game on $T_\phi(C_\omega)$ where $\forall$ plays constants, $\exists$ is "constant conservative"
  4. $\exists$ wins ψ-game on $T_\phi(C_m)$ where $\forall$ plays constants, $\exists$ is "constant conservative"
- Proof (3) $\Rightarrow$ (4): $\exists$ winning strategy for (3) is an $\exists$ winning strategy for (4)!

Proof

- Theorem: Let ϕ, ψ be conjunctive positive, let ψ have m universal variables. The following are equivalent:
  1. $\phi \models \psi$
  2. $\exists$ wins ψ-game on $T_\phi(C_\omega)$ where $\forall$ plays constants
  3. $\exists$ wins ψ-game on $T_\phi(C_\omega)$ where $\forall$ plays constants, $\exists$ is "constant conservative"
  4. $\exists$ wins ψ-game on $T_\phi(C_m)$ where $\forall$ plays constants, $\exists$ is "constant conservative"
- Proof (4) $\Rightarrow$ (1):
  - Suppose $\mathbf{B} \models \phi$. Want to show: $\mathbf{B} \models \psi$.
  - As $\forall$ vars. in ψ are being set to, say, $b_1, b_2, \ldots$...
  - We use values to construct a hom. $T_\phi(C_m) \rightarrow \mathbf{B}$ sending $c_1 \rightarrow b_1$, $c_2 \rightarrow b_2$, ...
  - From constant conservative strategy, this hom. tells us where to send $\exists$ vars. of ψ

Getting to decidability...

$\phi \models \psi$ equivalent to:

4. $\exists$ wins ψ-game on $T_\phi(C_m)$ where $\forall$ plays constants, $\exists$ is "constant conservative"

- Can show that if (4) is true, we can bound the "rank" of the terms that $\exists$ needs to use to win
- Leads to an algorithm: check if ψ is true on a bounded-rank substructure of $T_\phi(C_m)$

CSP over a relational structure

- CSP(**A**): set of all pp sentences $(\exists, \wedge)$ true over **A**
- Research question: classify the complexity of CSP(**A**) for various **A**
- Places into a uniform framework various problems such as:
  - 3-SAT
  - 2-SAT
  - other boolean sat. problems (e.g. 1-in-3 SAT)
  - k-Coloring ($\mathbf{A} = k$-clique)
  - Solving systems of equations

CSP: Algebraic Approach

- Algebraic approach to studying CSP($\mathbf{A}$) - Jeavons & co-authors
- Goal here: intuitively convey basic ideas
- Idea: relational structure $\mathbf{A} \longrightarrow$ set of operations
- Def: An op. $f : A^k \to A$ is a *polymorphism* of structure $\mathbf{A}$ if in every relation $R^{\mathbf{A}}$, the tuples are closed under coordinate-wise action of f
- Equivalent Def: An op. $f : A^k \to A$ is a polymorphism of $\mathbf{A}$ if it is a homomorphism $\mathbf{A}^k \to \mathbf{A}$

CSP: Algebraic Approach

- 3 steps
- Def: $(\exists\wedge)$-def$(\mathbf{A})$ denotes the set of pp-definable relations over structure $\mathbf{A}$

1. Theorem: If $(\exists\wedge)\text{-def}(\mathbf{A}) \subseteq (\exists\wedge)\text{-def}(\mathbf{B})$ then $\text{CSP}(\mathbf{A}) \leq_p^m \text{CSP}(\mathbf{B})$
2. Theorem (Geiger, Bodnarchuk et al.): $(\exists\wedge)\text{-def}(\mathbf{A}) = \text{Inv}(\text{Pol}(\mathbf{A}))$
   - $\text{Pol}(\mathbf{A})$ = set of polymorphisms of $\mathbf{A}$
   - $\text{Inv}(F)$ = all relations having F as polymorphisms
3. Theorem: If $\text{Pol}(\mathbf{B}) \subseteq \text{Pol}(\mathbf{A})$ then $\text{CSP}(\mathbf{A}) \leq_p^m \text{CSP}(\mathbf{B})$

Quantified CSP

- QCSP($\mathbf{A}$): set of all cp sentences ($\forall, \exists, \wedge$) true over $\mathbf{A}$
- PSPACE-complete in general
- Known examples of PSPACE-complete problems can be viewed as problems QCSP($\mathbf{A}$)
  e.g. Quantified 3-SAT

QCSP: Algebraic Approach

- Each of the three statements in previous slide hold for QCSP($\cdot$) in place of CSP($\cdot$)
- e.g., we have:
  1. Theorem: If $(\exists\wedge)$-def$(\mathbf{A}) \subseteq (\exists\wedge)$-def$(\mathbf{B})$ then QCSP$(\mathbf{A}) \leq_p^m$ QCSP$(\mathbf{B})$
- But, we may observe that the following also holds:
  1. Theorem: If $(\forall\exists\wedge)$-def$(\mathbf{A}) \subseteq (\forall\exists\wedge)$-def$(\mathbf{B})$ then QCSP$(\mathbf{A}) \leq_p^m$ QCSP$(\mathbf{B})$
- Why do these hold? Proof: to reduce from QCSP($\mathbf{A}$) to QCSP($\mathbf{B}$), replace each relation in $\mathbf{A}$ with a definition in $\mathbf{B}$

QCSP: Algebraic Approach

- What about the second ingredient?
  2. Theorem (Geiger, Bodnarchuk et al.):
     $(\exists\wedge)\text{-def}(\mathbf{A}) = \text{Inv}(\text{Pol}(\mathbf{A}))$
- We have the following "analog":
  2. Theorem (Börner et al. '03):
     $(\forall\exists\wedge)\text{-def}(\mathbf{A}) = \text{Inv}(\text{SurjPol}(\mathbf{A}))$

  $\text{SurjPol}(\mathbf{A})$ = surjective polymorphisms of $\mathbf{A}$

QCSP: Algebraic Approach

- Putting these together...
- In CSP, we had:
  3. Theorem: If Pol(**B**) $\subseteq$ Pol(**A**) then CSP(**A**) $\leq_p^m$ CSP(**B**)
- In QCSP, we have:
  3. Theorem: If SurjPol(**B**) $\subseteq$ SurjPol(**A**) then QCSP(**A**) $\leq_p^m$ QCSP(**B**)

QCSP: Algebraic Approach

- Set of operations $\text{Pol}(\mathbf{A})$ always a *clone*:
  contains all projections, closed under composition
- Parental advisory warning: $\text{SurjPol}(\mathbf{A})$ is (in general) not a clone!
  not always closed under composition
  - Understanding how to work with sets of operations $\text{SurjPol}(\mathbf{A})$ an open research issue

QCSP: Bounded Alternation

- What about the bounded alternation QCSP?
- e.g. Σ_{15}-QCSP($\mathbf{A}$)
- For "hard"/"expressive" $\mathbf{A}$, these characterize (are complete for) levels of polynomial hierarchy, e.g. Σ_{15}^{p}

QCSP: Bounded Alternation

- Algebraic approach gives us tools for discussing "expressiveness" of various $\mathbf{A}$
- To prove hardness results in the case of bd. alternation, would like to show:
  1'. Theorem: If $(\forall\exists\wedge)$-def$(\mathbf{A}) \subseteq (\forall\exists\wedge)$-def$(\mathbf{B})$ then Σ_{15}-QCSP$(\mathbf{A}) \leq_p^m \Sigma_{15}$-QCSP$(\mathbf{B})$
- Proof before does *not* work!
  If we substitute relations in QCSP$(\mathbf{A})$ with $(\forall, \exists, \wedge)$ definitions in $\mathbf{B}$, may increase quantifier prefix.
- Note: can be shown that $(\forall\exists\wedge)$-def$(\mathbf{A}) = \Pi_2$-$(\forall\exists\wedge)$-def$(\mathbf{A})$ (Chen '06)

QCSP: Bounded Alternation

- Theorem (Chen '06): Fix a quant. prefix $P \in \{\Pi_2, \Sigma_3, \Pi_4, \ldots\}$.

  If $(\forall\exists\wedge)$-def($\mathbf{A}$) $\subseteq$ $(\forall\exists\wedge)$-def($\mathbf{B}$) then P-QCSP($\mathbf{A}$) $\leq_p^m$ P-QCSP($\mathbf{B}$)
- Note: relies on finiteness of structures

Open Questions

- For the last two containment problems, we showed decidability; what is complexity of these problems?
  CSP: all problems NP-complete
- We have seen some recurring themes: collapses to Π_2 fragments, relevance of surjective homomorphisms
  Can any of these results be (partially) unified?
- What about entailment $\phi \models \psi$ on finite models?
  - For pp-sentences, this is true in the finite iff true in general

Open Questions

- Does result on QCSP($\mathbf{A}$) $\subseteq$ QCSP($\mathbf{B}$) give any insight into QCSP($\mathbf{A}$) complexity?
  Note: it is known that CSP($\mathbf{A}$) $=$ CSP($\mathbf{B}$) if and only if *cores* of $\mathbf{A}, \mathbf{B}$ are isomorphic
- Does result on entailment $\phi \models \psi$ give any insight into QCSP complexity?
  - Two pp-sentences ϕ, ψ are logically equivalent iff "have isomorphic cores";
    this has given insight into "left-hand side" CSP complexity, with links to bounded treewidth (Dalmau, Kolaitis, Vardi '02)