Hill, Hopkins and Ravenel's Detection Theorem

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The role of the Detection Theorem

There is a ring spectrum Ω so that

- Ω is 256 periodic (Periodicity Theorem)
- $\pi_k(\Omega) = 0$ for $-3 \leq k \leq -1$ (Gap Theorem)

and so that

Theorem (Detection)

If $\theta_j \in \pi_{2^{j+1}-2}$ exists then it has non-zero image in $\pi_{2^{j+1}-2}(\Omega)$.

Our plan:

1. Describe E_2 representatives for θ_j.
2. Construct spectrum Ω as $\Omega_{\mathbb{O}}^{hC_8}$.
3. Construct map of spectral sequences from ANSS for S^0 to HFPSS for $\pi_*\Omega_{\mathbb{O}}^{hC_8}$.
4. Construct map of C_8-modules $\pi_*^u\Omega_{\mathbb{O}} \to R_*$ for an R_* explicit enough to calculate $H^*(C_8; R_*)$.
5. Make calculations to show elements of E_2 term of ANSS that might detect θ_j have non-zero image in $H^*(C_8; R_*)$.

1: E_2 representatives

Recall Browder's theorem that θ_j exists if and only if the class

$$h_j^2 = [\xi_1^{2^j} | \xi_1^{2^j}] \in \mathrm{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbf{Z}/(2))$$

is a permanent cycle and detects θ_j.

We want to understand possible representatives in the ANSS also. The map of spectra $BP \to H\mathbf{Z}/(2)$ gives a map of Adams towers, which cannot decrease filtration. So if θ_j exists, then it is detected in the ANSS by an element of one of

$$\mathrm{Ext}_{BP}^{0,2^{j+1}-2}(BP_*), \mathrm{Ext}_{BP}^{1,2^{j+1}-1}(BP_*), \mathrm{Ext}_{BP}^{2,2^{j+1}}(BP_*).$$

The first two groups are 0. The third group has a basis (Shimomura, Mitchell):

$$\alpha_1 \cdot \alpha_{2^j-1}, \beta_{2^{j-1}/2^{j-1}}, \beta_{3\cdot 2^{j-3}/2^{j-3}}, \beta_{11\cdot 2^{j-5}/2^{j-5}}, \ldots$$

($\beta_{c(j,k)/2^{j-2k-1}}$ for $0 \le k < j/2$, $c(j,k) = 2^{j-1-2k}(1+2^{2k+1})/3$).

The αs and βs

The element $\alpha_s \in \mathrm{Ext}_{BP}^{1,2s}(BP_*)$ is constructed as follows. First recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \ldots]$, and is a BP_*BP-comodule.

- $v_1^s \in \mathrm{Ext}_{BP}^{0,2s}(BP_*/(2))$.
- Using the long exact sequence in Ext from

$$0 \to BP_* \xrightarrow{\cdot 2} BP_* \to BP_*/(2) \to 0$$

define $\alpha_s = \delta(v_1^s) \in \mathrm{Ext}_{BP}^{1,2s}(BP_*)$. This is $sv_1^{s-1}[t_1]$ (mod 4).

The βs are constructed similarly, but with two connecting homomorphisms. For suitable t (depending on s), $v_2^s \in \mathrm{Ext}_{BP}^{0,6s}(BP_*/(2, v_1^t))$. Then using the short exact sequence

$$0 \to BP_*/(2) \xrightarrow{v_1^t} BP_*/(2) \to BP_*/(2, v_1^t) \to 0$$

one has $\delta(v_2^s) \in \mathrm{Ext}_{BP}^{1,6s-2t}(BP_*/(2))$. Then using

$$0 \to BP_* \xrightarrow{\cdot 2} BP_* \to BP_*/(2) \to 0$$

one has $\delta(\delta(v_2^s)) \in \mathrm{Ext}_{BP}^{2,6s-2t}(BP_*)$.

One can calculate a formula for such elements in the cobar complex. The most interesting case is

$$\beta_{2^{j-1}/2^{j-1}} = \delta(\delta(v_2^{2^{j-1}})) = \sum_{k=1}^{2^j-1} \frac{1}{2}\binom{2^j}{k}[t_1^k|t_1^{2^j-k}].$$

Note that all the binomical coefficients are divisible by 2. Also, all but $\binom{2^j}{2^{j-1}}$ are divisible by 4.

In the reduction map from the ANSS to the ASS, $t_1 \mapsto \xi_1^2$. So the image of $\beta_{2^{j-1}/2^{j-1}}$ is $[\xi^{2^j}|\xi^{2^j}] = h_j^2$!
One can argue similarly that all other generators of $\mathrm{Ext}_{BP}^{2,2^{j+1}}(BP_*)$ reduce to 0 in $\mathrm{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbf{Z}/(2))$.
We deduce that:

Any class detecting θ_j in the ANSS is $\beta_{2^{j-1}/2^{j-1}}+$ a linear combination of other basis elements.

To prove the detection theorem we now need to show that if $\beta_{2^{j-1}/2^{j-1}} + x$ is a permanent cycle in the ANSS then the class it represents maps to a nonzero element of Ω.
We still need to

- Construct Ω.
- Construct appropriate map of spectral sequences.
- Construct an appropriate map of C_8-modules.
- Do the relevant calculations.

2: Constructing $\Omega_{\mathbb{O}}$

We use the norm construction

$$N_{C_2}^{C_8} : C_2\text{-Spectra} \to C_8\text{-Spectra}$$

Among the properties we need are

1. $i_{C_2}^*(N_{C_2}^{C_8}(X)) = X \wedge X \wedge X \wedge X = X^{(4)}$.
2. $N_{C_2}^{C_8}(S^{\rho_2}) = S^{\rho_8}$.
3. $N_{C_2}^{C_8}$ is left adjoint to $i_{C_2}^*$, giving an adjunction map $N_{C_2}^{C_8}(i_{C_2}^*(X)) \to X$ for a C_8 spectrum X.

Then we define $\Omega_{\mathbb{O}}$ by

$$\Omega_{\mathbb{O}} = D^{-1} N_{C_2}^{C_8}(MU_{\mathbf{R}})$$

where

$$D = N_{C_2}^{C_8}(\overline{r}_{15}^{C_2}\overline{r}_3^{C_4}\overline{r}_1^{C_8})$$

2: Constructing $\Omega_{\mathbb{O}}$ and Ω

Recall that $\bar{r}_i^G : S^{i\rho_2} \to i_{C_2}^* N_{C_2}^{C_8}(MU_{\mathbb{R}})$ and so when we norm up we have a map from $S^{19\rho_8}$ to $N_{C_2}^{C_8}(MU_{\mathbf{R}})$, using various adjunctions.

We will justify the choice of D presently, but we won't actually do the relevant calculation.

Finally, we can say what Ω is:

$$\Omega = \Omega_{\mathbb{O}}^{hC_8}.$$

(Remark: The Gap Theorem is actually about Ω^{C_8}. So to prove the K.I. 1 theorem, one must also show that $\Omega_{\mathbb{O}}^{hC_8} = \Omega_{\mathbb{O}}^{C_8}$)

3: ANSS to HFPSS

Notation: write $(X)^{(G)} := \mathrm{Maps}(G, X)$.

One way to think about the ANSS is to consider the pair of spectra, $(MU_{\mathbb{R}}, MU_{\mathbb{R}} \wedge MU_{\mathbb{R}})$ and the C_2-equivariant structure maps making the (underlying) homotopy groups into a Hopf algebroid.

Using this data, we can make a cosimplicial spectrum

$$MU_{\mathbb{R}} \to MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \to MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \to \dots$$

The associated spectral sequence is the ANSS.

Similarly, consider the pair of spectra $(\Omega_{\mathbb{O}}, (\Omega_{\mathbb{O}})^{(C_8)})$ with

$$\eta_L(x) = \{g \mapsto x\} \text{ and } \eta_R(x) = \{g \mapsto gx\}$$

With this data we can also make a cosimplicial spectrum

$$\Omega_{\mathbb{O}} \to (\Omega_{\mathbb{O}})^{(C_8)} \to (\Omega_{\mathbb{O}})^{(C_8 \times C_8)} \to \dots$$

The associated spectral sequence is the HFPSS.

3: ANSS to HFPSS

We wish to map the first cosimplicial spectrum to the second. We do this by mapping $MU_{\mathbb{R}} \to MU_{\mathbb{R}}^{(4)}$ to the first factor, and then mapping

$$(C_8)_+ \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \to MU_{\mathbb{R}}^{(4)}$$

using the various orientations of $MU_{\mathbb{R}}^{(4)}$ and the isomorphisms between them. I.e., we map

$$\{e\}_+ \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \to MU_{\mathbb{R}}^{(4)}$$

to the first two factors, and then extend equivariantly using the C_8 action on C_8 on the domain, and on $N_{C_2}^{C_8}(MU_{\mathbb{R}})$ on the range. Then we taken the adjoint and invert D to get

$$MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \to (MU_{\mathbb{R}}^{(4)})^{(C_8)} \to (\Omega_{\mathbb{O}})^{(C_8)}.$$

4: Map to a simpler C_8-module

Recall that if $F(x, y) \in A[[x, y]]$ is a formal group law, the n-series is $[n](x) = F(F(\dots(x, x)))$ ($n-1$ applications of F). If A is a Z_p algebra and F is p-typical this can be extended to a ring homomorphism

$$\mathbf{Z}_p \to \mathrm{End}(F).$$

If, in addition, A is a DVR (char. 0), with uniformizer π and residue field $\mathbf{F}_{p^s}$, then given a log series over the field of fractions of A satisfying

$$\log(x) = x + \pi_{-1} \log(x^q)$$

there is a formal group law over A

$$F(x, y) = \log^{-1}(\log(x) + \log(y))$$

and a ring homomorphism

$$A \to \mathrm{End}(F)$$

given by

$$a \mapsto [a](x) = \log^{-1}(a \ \log(x)).$$

4: Map to a simpler C_8-module

This ring homomorphism extends the one from $\mathbf{Z}_p$ and such an F is called a *formal A-module*.

We take $A = \mathbf{Z}_2[\zeta]$ where ζ is a primitive 8th root of 1. Here $\pi = \zeta - 1$, and $(\pi^4) = (2)$. We introduce a polynomial variable w of degree 2 to keep things homogeneous, so $R_* = A[w^{\pm 1}]$ and write

$$\log_F(x) = \sum_{k=0}^{\infty} \frac{x^{2^k}}{\pi^k} w^{2^k - 1}$$

which satisfies the condition on the previous slide.

$$F(x, y) = \log_F^{-1}(\log_F(x) + \log_F(y))$$

is a formal A-module over $R_* = A[w^{\pm 1}]$ (and by forgetting, a p-typical FGL). It has an endomorphism $[\zeta](x)$ which is a primitive 8th root of 1 in $\mathrm{End}(F)$.

Remark: From the log series, this looks like height 1 if one thinks of π as analagous to p. But π^4 is a unit times p, so this is height 4.

4: $\pi^u_*(\Omega_{\mathbb{O}}) \to R_*$

C_8 acts on R_* by the identity on A and $\gamma w = \zeta \cdot w$, extending to a ring homomorphism.

To map $\pi_* MU \to R_*$ we use the FGL F. We actually want to map $\pi^u_*(\Omega_{\mathbb{O}})$ to R_* (equivariantly).
First note that a map:

$$\pi^u_* MU^{(4)} \to R_* \tag{1}$$

is given by four FGLs and three strict isomorphisms. We use

$$f(x) = [\zeta](\zeta^{-1} x)$$

as all of the strict isomorphisms, and F as the first FGL (so f determines the other three FGLs).
This gives the map (1).

4: $\pi_*^u(\Omega_{\mathbb{O}}) \to R_*$

A calculation using the definition of D and using the logarithm of F and one of the isomorphic FGLs over R_* shows that the image of D is a unit, and that smaller subscripts on the $\bar{r}_i$ would not have given a unit. This is part of what justifies that choice of D.

The image of possible K.I. 1 detectors in $H^*(C_8; R_*)$:

$H^*(C_8; R_*)$ is determined by the Hopf algebroid $(R_*, R_*^{(C_8)})$ (the second ring is functions from C_8 to R_*).

The map from MU_*MU to $R_*^{(C^8)}$ is determined by the map from $BP_*BP = BP_*[t_1, t_2, \ldots]$. The images of the t_i are determined by the equation

$$f(x) = [\zeta](\zeta^{-1}x) = \sum_{i\geq 0}^{F} t_i(\zeta)(x)^{2^i} \qquad (2)$$

5: The image of possible K.I. 1 detectors in $H^*(C_8; R_*)$

Working from the equation on the previous slide:

Apply $\log_F$: $\zeta \log_F(\zeta^{-1}x) = \zeta \sum_{k=0}^{\infty} \frac{(\zeta^{-1}x)^{2^k}}{\pi^k} w^{2^k-1} = \sum_{i \geq 0} \log(t_i(\zeta) x^{2^i}$

Extract coefficient of x^{2^n}: $$\zeta \frac{w^{2^n-1}\zeta^{-2^n}}{\pi^n} = \sum_{i+j=n} \frac{w^{2^j-1} t_i(\zeta)^{2^j}}{\pi^j}$$

At $n = 0$: $$\zeta \cdot \zeta^{-1} = 1 \cdot t_0(\zeta), \text{ so } t_0 = 1.$$

At $n = 1$: $$\zeta \frac{w\zeta^{-2}}{\pi} = \frac{w t_0(\zeta)^2}{\pi} + \frac{1 \cdot t_1(\zeta)}{1}.$$

Solving: $t_1(\zeta) = \frac{w}{\zeta}$. (This is a unit in R_2).

We want to use this calculation to check that in the composite

$$\mathrm{Ext}_{BP}^{2,2^{j+1}}(BP_*) \to H^2(C_8; \Omega_{\mathbb{O}}) \to H^2(C_8; R_*) \tag{3}$$

the element $\beta_{2^{j-1}/2^{j-1}}$ goes to a non-zero element, and other basis elements go to 0.

5: The image of possible K.I. 1 detectors in $H^*(C_8; R_*)$

The groups $H^*(C_8; R_*)$ can be calculated in the usual way (one topological degree at a time) from the complex

$$R_* \xrightarrow{\gamma-1} R_* \xrightarrow{1+\gamma+\gamma^2+\cdots+\gamma^7} R_* \to \dots$$

Such a calculation will give us that ($j \geq 3$):

$$H^1(C_8; R_{2^j}) = 0 \text{ and } H^1(C_8; R_{2^j}/(2)) \cong w^{2^{j-1}} A/(2) \qquad (4)$$

Using this, consider the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ext}_{BP}^{1,2^{j+1}}(BP_*) & \longrightarrow & \mathrm{Ext}_{BP}^{1,2^{j+1}}(BP_*/(2)) & \xrightarrow[\delta]{} & \mathrm{Ext}_{BP}^{2,2^{j+1}}(BP_*) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(C_8; R_{2^{j+1}}) & \longrightarrow & H^1(C_8; R_{2^{j+1}}/(2)) & \xrightarrow[\delta]{} & H^2(C_8; R_{2^{j+1}}) \\
= \downarrow & & \downarrow \cong & & \downarrow \\
0 & \longrightarrow & w^{2^j} A/(2) & \longrightarrow & w^{2^j} A/(8)
\end{array}$$

If we start with $[t_1^{2^j}]$ in the center top, its image is $[w^{2^j}] \neq 0$!

5: The image of possible K.I. 1 detectors in $H^*(C_8; R_*)$

What about the rest? R_* has a valuation determined by $||\pi|| = 1/4$ (and thus $||2|| = 1$). One can calculate using the logarithms of the FGLs over BP_* and R_* that

$$v_1 \mapsto \pi^3 \cdot \text{unit}$$

$$v_2 \mapsto \pi^2 \cdot \text{unit}$$

$$v_3 \mapsto \pi \cdot \text{unit}$$

$$v_4 \mapsto \text{unit}$$

Put a valuation on BP_*BP by $||t_n|| = ||v_i|| = 0$ for all n, $i \geq 4$,and $||2|| = 1$, $||v_1|| = \frac{3}{4}$, $||v_2|| = \frac{1}{2}$, $||v_3|| = \frac{1}{4}$.

The map (3) doesn't lower valuation, and one can check that the other elements of $\mathrm{Ext}_{BP}^{2,2^{j+1}}(BP_*)$ have valuations ≥ 3. So their images are divisible by 8, and thus 0 in $H^2(C_8; R_{2^j})$.

Conclusion

Recall:

$$\begin{array}{ccccc} \mathrm{Ext}^*_{BP}(BP_*) & \longrightarrow & H^*(C_8; \pi_*\Omega_{\mathbb{O}}) & \longrightarrow & H^*(C_8; R_*) \\ \Downarrow\big\downarrow & & \big\downarrow\Downarrow & & \\ \pi_*(S^0_{(2)}) & \longrightarrow & \pi_*(\Omega) & & \end{array}$$

Suppose an element of the ANSS survives to θ_j for $j \geq 3$. Then (since the map to the HFPSS is a map of spectral sequences), its image is a permanent cycle in the HFPSS, so gives an element of $\pi_{2^{j+1}-2}(\Omega)$.

The element of the E_2-term of the HFPSS maps to a nonzero element in $H^*(C_8; R_*)$, so it is non-zero on the E_2 term, and thus survives the spectral sequence to give a non-zero element of $\pi_{2^{j+1}-2}(\Omega)$.