Remakes of Browder's paper on the Kervaire invariant

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1. Browder's result

Let M^{2n} ($n > 0$) be a closed manifold equipped with a framing t. This framing determines a map

$$\mathsf{q}_t : \mathsf{H}^n M = \mathsf{H}^n(M; \mathbb{Z}/2) \to \mathbb{Z}/2$$

such that one has

$$\mathsf{q}_t(x+y) = \mathsf{q}_t(x) + \mathsf{q}_t(y) + \langle\, x \smile y \,,\, [M]\,\rangle$$

for all x and y in $\mathsf{H}^n M$. In other words q_t is a quadratic form (the *Kervaire quadratic form*) associated to the non-degenerate symmetric bilinear form

$$\mathsf{H}^n M \times \mathsf{H}^n M \to \mathbb{Z}/2 \ \ , \ \ (x,y) \mapsto \langle\, x \smile y \,,\, [M]\,\rangle \ \ .$$

The *Kervaire invariant* of (M,t), let us say $\kappa(M,t)$, is the Arf invariant of q_t. One shows that $\kappa(M,t)$ depends only on the class of (M,t) in the cobordism group $\Omega^{\mathsf{fr}}_{2n}$.

Recall of Browder's definition of Kervaire's quadratic form

Browder considers the map $\mathsf{Q}_t : \mathsf{H}^n M \to \pi_{2n}(\mathsf{S} \wedge \mathsf{K}_n)$ ($\mathsf{K}_n := \mathsf{K}_n(\mathbb{Z}/2, n)$) sending a class x to the class of the composed map

$$\mathsf{S}^{2n} \xrightarrow{\text{Thom map}} \mathsf{Th}(\nu_M) \underset{\sim}{\xrightarrow{t}} \mathsf{S} \wedge M_+ \xrightarrow{\mathsf{S}\wedge x} \mathsf{S} \wedge \mathsf{K}_n$$

and defines $\mathsf{q}_t : \mathsf{H}^n M \to \mathbb{Z}/2$ by the composition (the composed map $\pi_{2n}(\mathsf{S} \wedge \mathsf{K}_n) \to \mathbb{Z}/2$ which appears below is in fact an isomorphism)

$$\mathsf{H}^n M \xrightarrow{\mathsf{Q}_t} \pi_{2n}(\mathsf{S} \wedge \mathsf{K}_n) = \mathsf{F}^1 \pi_{2n}(\mathsf{S} \wedge \mathsf{K}_n)$$
$$\to \mathsf{Ext}^1_{\mathsf{A}}(\mathsf{H}^*(\mathsf{S} \wedge \mathsf{K}_n), \mathbb{Z}/2) \xrightarrow{\gamma := \mathsf{Sq}^{n+1} \otimes_{\mathsf{A}} \iota_n} \mathbb{Z}/2$$

(A denotes the mod 2 Steenrod algebra, the Adams filtration for the Adams spectral sequence associated to mod 2 cohomology is generically denoted by $\mathsf{F}^0 \supset \mathsf{F}^1 \supset \mathsf{F}^2 \supset \ldots$ and γ is the element in $\mathsf{Tor}^{\mathsf{A}}_1(\mathbb{Z}/2, \mathsf{H}^*(\mathsf{S} \wedge \mathsf{K}_n)))$ corresponding to the unstability relation $\mathsf{Sq}^{n+1}\, \iota_n = 0$).

TH 1 (Browder, 1969). *Let χ be the canonical anti-automorphism of* A. *Then there exists an element* b_2 *in* $\mathrm{Tor}_2^{\mathsf{A}}(\mathbb{Z}/2,\mathbb{Z}/2)$ *represented by a relation in* A *of the form*

$$\mathrm{Sq}^{n+1}(\chi\mathrm{Sq}^{n+1}) + \sum_{i=1}^{n} \mathrm{Sq}^{n+1-i} a_i = 0$$

such that the Kervaire invariant $\kappa : \Omega_{2n}^{\mathrm{fr}} \to \mathbb{Z}/2$ *coincides with the following composed map*

$$\Omega_{2n}^{\mathrm{fr}} \cong \pi_{2n}\mathrm{S} = \mathrm{F}^2\pi_{2n}\mathrm{S} \longrightarrow \mathrm{Ext}_{\mathsf{A}}^2(\mathbb{Z}/2,\mathbb{Z}/2) \xrightarrow{\mathsf{b}_2} \mathbb{Z}/2 \quad .$$

Rm. A canonical form for the relation defining b_2 is the following one:

$$\sum_{i=0}^{n} \binom{n+1+i}{i} \mathrm{Sq}^{n+1-i}\chi\mathrm{Sq}^{n+1-i} = 0 \quad .$$

COR. *The Kervaire invariant is trivial if $n+1$ is not a power of 2. If $n+1$ is of the form 2^k then it is non-trivial if and only if in the Adams spectral sequence (the same than the one evoked above) the element h_k^2 in E^2 persists to E^∞.*

. . .

Fourty years later we know, thanks to Hill, Hopkins and Ravenel, that h_k^2 does not persist to E^∞ for $k \geq 7$ (we knew it persisted for $k \leq 5$).

2. Kervaire invariant and manifolds with boundary

TH 2 (L. 1981). *Let* b_1 *be the element in* $\mathrm{Tor}_1^{A}(\mathbb{Z}/2, H^*(MO/S))$ *represented by the following relation in* H^*MO*:*

$$\sum_{i=0}^{n} Sq^{n+1-i}(U \smile v_i v_{n+1}) = 0$$

(observe that $H^*(MO/S)$ *identifies with the kernel of the restriction* $H^*MO \to H^*S = \mathbb{Z}/2$*). Then the following diagram*

$$\begin{array}{ccc} F^1\pi_{2n+1}(MO/S) & \longrightarrow & \mathrm{Ext}_A^1(H^*(MO/S), \mathbb{Z}/2) \\ \downarrow \cong & & \downarrow b_1 \\ \pi_{2n}S & \xrightarrow{\kappa} & \mathbb{Z}/2 \end{array}$$

is commutative.

About the relation b_1

- $\chi \mathsf{Sq}^k \mathsf{U} := \mathsf{U} \smile \mathsf{v}_k$

- From the very definition of χ one has

$$\sum_{i+j=k} \mathsf{Sq}^i((\mathsf{U} \smile \mathsf{v}_j) \smile x) = \mathsf{U} \smile \mathsf{Sq}^k x$$

for any x in $\mathsf{H}^*\mathsf{BO}$.

- We take $k = n + 1$ and $x = \mathsf{v}_{n+1}$ then $\mathsf{U} \smile \mathsf{v}_{n+1}^2$ appears on both sides. Hence the relation.

Theorems 1 and 2 are equivalent because b_2 and b_1 correspond to each other *via* the isomorphism

$$\mathsf{Tor}_2^{\mathsf{A}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathsf{Tor}_1^{\mathsf{A}}(\mathbb{Z}/2, \mathsf{H}^*(\mathsf{MO}/\mathsf{S})) \quad .$$

Three of the ingredients coming into the proof of Theorem 2

- First the interpretation $\pi_{2n+1}(\mathsf{MO}/\mathsf{S}) \cong \Omega_{2n+1}^{\mathsf{O},\mathsf{fr}}$.
- Then the two following propositions.

PROP A. *Let E be a finite-dimensional $\mathbb{Z}/2$-vector space equipped with a non-degenerate quadratic form* q. *Let I be a subspace of E with $I = I^{\perp}$ and u be an element in E such that one has* $\mathsf{q}(x) = u.x$ *for all x in I. Then:*

$$\mathsf{Arf}(\mathsf{q}) = \mathsf{q}(u) \quad .$$

Proof of Proposition A

1st case: $\mathsf{q}(I) = 0$. Then $E \approx \mathsf{H}(I)$ ($\mathsf{H}(I)$ denotes the hyperbolic quadratic space on I), $u \in I$ and $\mathsf{q}(u) = 0$. $\square$

2nd case: $\mathsf{q}(I) \neq 0$. Then $\mathsf{q}_{|I}$ is a non-zero linear form; let J be its kernel and v be an element in I with $\mathsf{q}(v) = 1$. Let $F \subset E$ be the subspace generated by u and v. The equality

$$\begin{bmatrix} u.u & u.v \\ v.u & v.v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

implies $\dim F = 2$ and $E = F \oplus F^{\perp} \approx F \oplus \mathsf{H}(J)$.

If $\mathsf{q}(u) = 0$ then $F \approx \mathsf{H}(\mathbb{Z}/2)$; if $\mathsf{q}(u) = 1$ then F is isomorphic to $(\mathbb{Z}/2)^2$ equipped with the quadratic form $(x, y) \mapsto x^2 + xy + y^2$.

$\square$

Now let N^{2n+1} be a compact manifold whose boundary is M. Let us take $E = \mathsf{H}^n M$ and $I = \mathrm{im}(\mathsf{H}^n N \to \mathsf{H}^n M)$. The value of the Kervaire quadratic form on I is given by the following formula:

PROP B. *For all y in $\mathsf{H}^n N$ one has:*

$$\mathsf{q}_t(y|_M) = \langle\, \mathsf{v}_{n+1}(\nu_N, t) \smile y\,,\, [N]\,\rangle \quad ,$$

$\mathsf{v}_{n+1}(\nu_N, t)$ *denoting the relative Wu class defined in* $\mathsf{H}^{n+1}(N, M)$ *using the framing* t.

Proof of Proposition B

The data (N, y, M, t) determine an element $[N, y, M, t]$ in $\pi_{2n+1}((\mathsf{MO}/\mathsf{S}) \wedge \mathsf{K}_n)$ whose image in $\pi_{2n}(\mathsf{S} \wedge \mathsf{K}_n)$ is $[M, t, y_{|M}]$.
One considers the exact sequence of A-modules

$$0 \to \mathsf{H}^*(\mathsf{MO}/\mathsf{S}) \otimes \bar{\mathsf{H}}^*\mathsf{K}_n \to \mathsf{H}^*\mathsf{MO} \otimes \bar{\mathsf{H}}^*\mathsf{K}_n \to \bar{\mathsf{H}}^*\mathsf{K}_n \to 0 \quad .$$

• The element ι_n in $\bar{\mathsf{H}}^*\mathsf{K}_n$ lifts to $\mathsf{U} \otimes \iota_n$ in $\mathsf{H}^*\mathsf{MO} \otimes \bar{\mathsf{H}}^*\mathsf{K}_n$; $\mathsf{Sq}^{n+1}(\mathsf{U} \otimes \iota_n)$ identifies with an element in $\mathsf{H}^*(\mathsf{MO}/\mathsf{S}) \otimes \bar{\mathsf{H}}^*\mathsf{K}_n$, let us say ω_1. The image of $[N, y, M, t]$ by the homomorphism

$$\pi_{2n+1}((\mathsf{MO}/\mathsf{S}) \wedge \mathsf{K}_n) \xrightarrow{\omega_1} \mathbb{Z}/2$$

is $\mathsf{q}_t(y_{|M})$.

• Let us denote by ω_2 the element $(\mathsf{U}\,\widehat{v}_{n+1}) \otimes \iota_n$ of $\mathsf{H}^*(\mathsf{MO}/\mathsf{S}) \otimes \bar{\mathsf{H}}^*\mathsf{K}_n$. The image of $[N, y, M, t]$ by the homomorphism

$$\pi_{2n+1}((\mathsf{MO}/\mathsf{S}) \wedge \mathsf{K}_n) \xrightarrow{\omega_2} \mathbb{Z}/2$$

is $\langle \mathsf{v}_{n+1}(\nu_N, t) \smile y \,,\, [N] \rangle$.

Now one checks that the images of the classes ω_1 and ω_2 in $\mathsf{Tor}_0^{\mathsf{A}}(\mathbb{Z}/2, \mathsf{H}^*(\mathsf{MO}/\mathsf{S}) \otimes \bar{\mathsf{H}}^*\mathsf{K}_n)$ coincide. $\square$

Observations

- Proposition B implies the invariance of $\mathsf{Arf}(\mathsf{q}_t)$ under framed cobordism.

- Proposition B implies $\mathsf{v}_{n+1}(\nu_N) = 0$. Therefore the class $\mathsf{v}_{n+1}(\nu_N, t)$ belongs to the image of the connecting homomorphism

$$\partial : \mathsf{H}^n M \longrightarrow \mathsf{H}^{n+1}(N, M) \quad ;$$

let u be a class in $\mathsf{H}^n M$ with $\partial u = \mathsf{v}_{n+1}(\nu_N, t)$ then one has

$$\mathsf{q}_t(x) = \langle\, u \smile x \,,\, [M] \,\rangle \quad ,$$

for all x in $I = \mathrm{im}(\mathsf{H}^n N \to \mathsf{H}^n M)$. Hence Proposition A gives

$$\mathsf{Arf}(\mathsf{q}_t) = \mathsf{q}_t(u) \quad .$$

Proof of Theorem 2

We are left with the task to link the class $[N, M, t]$ in $\mathsf{F}^1\Omega^{\mathsf{O,fr}}_{2n+1} \cong \mathsf{F}^1\pi_{2n+1}(\mathsf{MO}, \mathsf{S})$ with the class $\mathsf{Q}_t(u)$ thought of as a class in $\mathsf{F}^1\pi_{2n+1}(\mathsf{S}, \mathsf{S} \wedge \mathsf{K}_{n+}) = \pi_{2n+1}(\mathsf{S}, \mathsf{S} \wedge \mathsf{K}_{n+}) \cong \pi_{2n}(\mathsf{S} \wedge \mathsf{K}_n)$. This is done by thinking of $N \times [0,1]$ as a cobordism between N and $M \times [0,1]$, equipped with some suitable normal structure (the "half-sock cobordism").

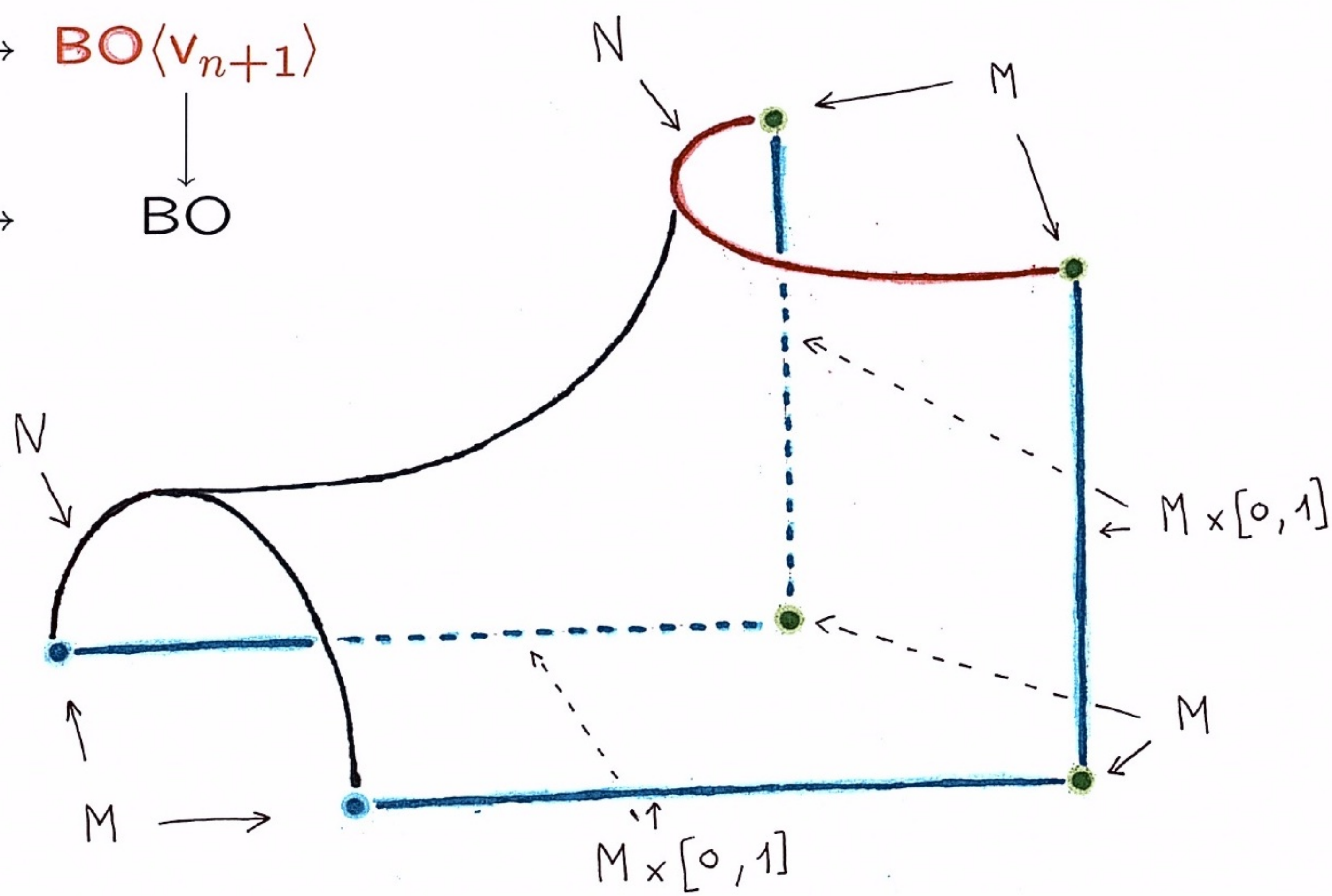
Following Browder, one introduces the space $\mathsf{BO}\langle \mathsf{v}_{n+1} \rangle$ fibre of the map $\mathsf{v}_{n+1} : \mathsf{BO} \to \mathsf{K}_{n+1}$ and one looks at the commutative diagram of spaces

$$\begin{array}{ccc} \mathsf{K}_n & \longrightarrow & \mathsf{BO}\langle \mathsf{v}_{n+1} \rangle \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathsf{BO} \end{array}$$

and at the induced diagram of stable vector bundles.

The half-sock cobordism

$$\begin{array}{ccc} \mathbf{K}_n & \longrightarrow & \mathbf{BO}\langle \mathsf{v}_{n+1} \rangle \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{BO} \end{array}$$



The half-sock cobordism shows that the images of $[N,M,t]$ and $\mathsf{Q}_t(u)$ in $\mathsf{F}^1\pi_{2n+1}(\mathsf{MO},\mathsf{MO}\langle \mathsf{v}_{n+1}\rangle)$ (diagram below) coincide.

$$\begin{array}{ccccc} \mathsf{F}^1\pi_{2n+1}(\mathsf{MO},\mathsf{S}) & \longrightarrow & \mathsf{F}^1\pi_{2n+1}(\mathsf{MO},\mathsf{MO}\langle \mathsf{v}_{n+1}\rangle) & \longleftarrow & \mathsf{F}^1\pi_{2n+1}(\mathsf{S},\mathsf{S}\wedge\mathsf{K}_{n+}) \\ \downarrow{\scriptstyle \mathsf{b}_1} & & \downarrow{\scriptstyle \beta} & & {\scriptstyle \Sigma\gamma}\downarrow{\scriptstyle \cong} \\ \mathbb{Z}/2 & = & \mathbb{Z}/2 & = & \mathbb{Z}/2 \end{array}$$

Now let $\beta \in \mathsf{Tor}_1^{\mathsf{A}}(\mathbb{Z}/2,\mathsf{H}^*(\mathsf{MO},\mathsf{MO}\langle \mathsf{v}_{n+1}\rangle))$ be the element corresponding to the relation

$$\sum_{i=0}^{n} \mathsf{Sq}^{n+1-i}(\mathsf{U}\smile \mathsf{v}_i\widehat{\mathsf{v}}_{n+1}) = 0$$

in $\mathsf{H}^*(\mathsf{MO},\mathsf{MO}\langle \mathsf{v}_{n+1}\rangle)$, then β induces b_1 in $\mathsf{Tor}_1^{\mathsf{A}}(\mathbb{Z}/2,\mathsf{H}^*(\mathsf{MO},\mathsf{S}))$ and $\Sigma\gamma = \mathsf{Sq}^{n+1}\otimes_{\mathsf{A}}\Sigma\iota_n$ in $\mathsf{Tor}_1^{\mathsf{A}}(\mathbb{Z}/2,\mathsf{H}^*(\mathsf{S},\mathsf{S}\wedge\mathsf{K}_{n+}) = \Sigma\mathsf{H}^*(\mathsf{S}\wedge\mathsf{K}_n))$.

Eventually we do get

$$\langle \mathsf{b}_1,[N,M,t]\rangle = \mathsf{q}_t(u) = \kappa(M,t) \quad .$$

Q. E. D.

3. Kervaire invariant and manifolds with a corner of codimension 2

Now we climb further up in the MO-spectral sequence:

TH 3 (Miller-L.). *Let* b *be the element in* $\mathrm{H}^*(\mathrm{MO}/\mathrm{S} \wedge \mathrm{MO}/\mathrm{S})$ *whose image in* $\mathrm{H}^*(\mathrm{MO} \wedge \mathrm{MO})$ *is:*

$$\sum_{i=0}^{n} (\mathrm{U} \smile \mathrm{v}_{n+1-i}) \times (\mathrm{U} \smile \mathrm{v}_i \mathrm{v}_{n+1}) \quad .$$

Then the following diagram

$$\begin{array}{ccc} \pi_{2n+2}(\mathrm{MO}/\mathrm{S} \wedge \mathrm{MO}/\mathrm{S}) & \longrightarrow & \mathrm{H}_{2n+2}(\mathrm{MO}/\mathrm{S} \wedge \mathrm{MO}/\mathrm{S}) \\ \downarrow & & \downarrow \mathrm{b} \\ \pi_{2n}\mathrm{S} & \xrightarrow{\kappa} & \mathbb{Z}/2 \end{array}$$

is commutative.

Let b_0 be the class of b in $\mathrm{Tor}_0^A(\mathbb{Z}/2, H^*(MO/S \wedge MO/S))$.

Theorem 1 (Browder's theorem) and Theorem 3 are equivalent because the left vertical arrow of the diagram above is surjective and b_0 is the image of b_2 under the monomorphism

$$\mathrm{Tor}_2^A(\mathbb{Z}/2, \mathbb{Z}/2) \hookrightarrow \mathrm{Tor}_0^A(\mathbb{Z}/2, H^*(MO/S \wedge MO/S)) \quad .$$

Proof of Theorem 3

- $\pi_{2n+2}(\mathsf{MO/S} \wedge \mathsf{MO/S}) \cong \Omega_{2n+2}^{(\mathsf{O,fr})^2}$.

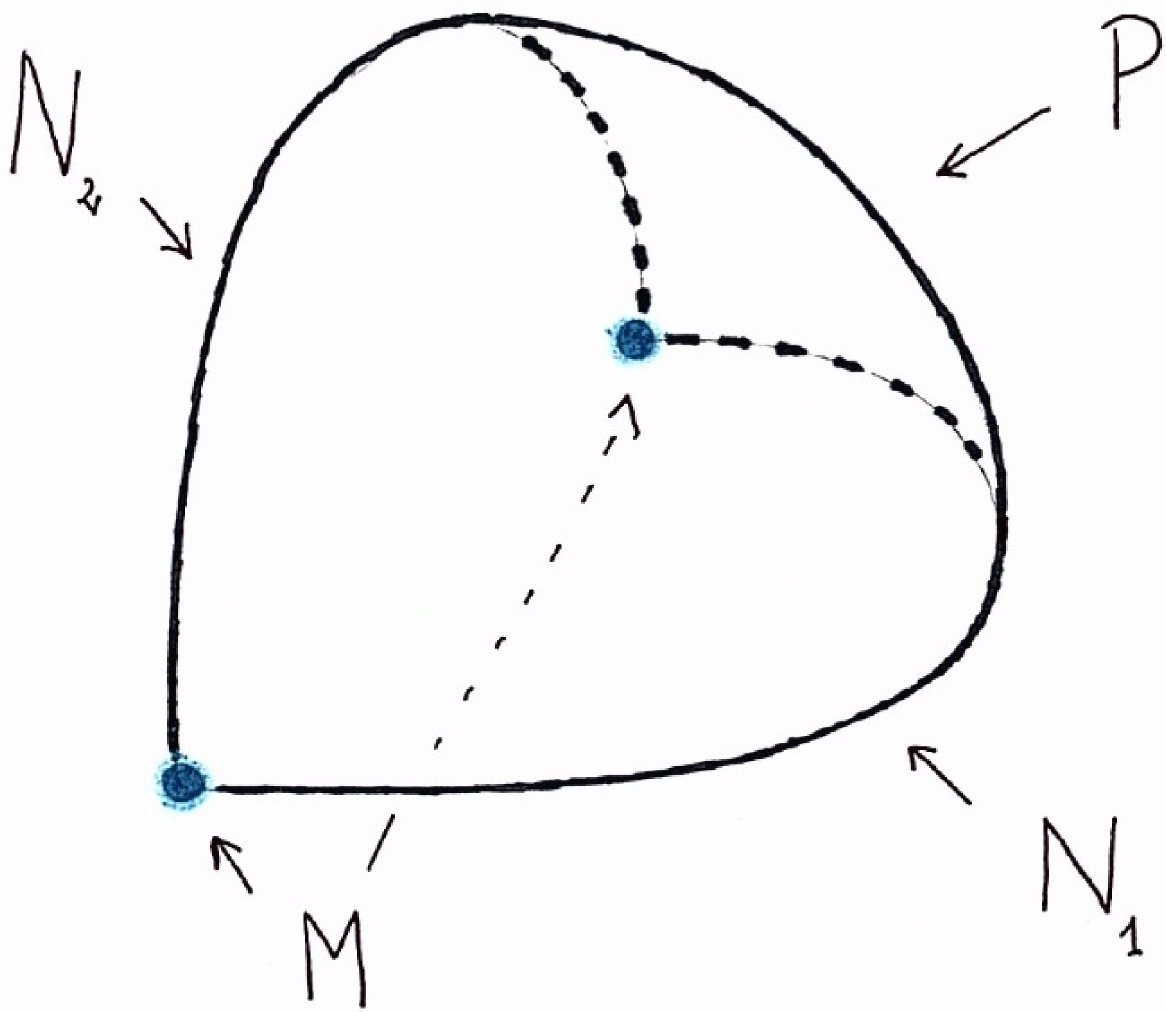
Let P be a $(2n+2)$-dimensional compact manifold with a corner of codimension 2. One denotes by M this corner and by N the boundary of P, N is the union of two manifolds with boundary, N_1 and N_2, whose intersection is M (their common boundary).

One assumes one has a splitting $\nu_P = \nu_1 \oplus \nu_2$ such that:

- there exists a trivialisation, let us say t_1, of the restriction of ν_1 to N_1;
- there exists a trivialisation, let us say t_2, of the restriction of ν_2 to N_2.

t_1 and t_2 determine a trivialisation of ν_M which is denoted by t.

The group $\Omega_{2n+2}^{(\mathsf{O,fr})^2}$ is the cobordism group of such Ps.



N_2
P
M
N_1

• Given a $(2n+2)$-dimensional compact manifold P with a corner of codimension 2, one produces a $\mathbb{Z}/2$-vector space E equipped with a non-degenerate symmetric bilinear form and a lagrangian subspace I. In this part there are no normal structures involved, we are just playing with Poincaré duality (that is to say the conjunction of Spanier-Whitehead duality and Thom isomorphism).

One sets $F = \mathsf{H}^n M$; F is equipped with the "intersection pairing".

One sets $G = \mathsf{H}^{n+1}(P, N_1) \oplus \mathsf{H}^{n+1}(P, N_2)$; one equips G with the symmetric bilinear form

$$(z_1, z_2).(z'_1, z'_2) = \langle\, z_1 \smile z'_2 + z'_1 \smile z_2\,,\ [P]\,\rangle$$

(G is canonically isomorphic to the hyperbolic space on $\mathsf{H}^{n+1}(P, N_1)$ because $\mathsf{H}^{n+1}(P, N_2)$ is Poincaré-dual of $\mathsf{H}_{n+1}(P, N_1)$).

One defines E as the orthogonal sum $F \oplus G$. One defines I as the image of the natural map

$$\mathsf{H}^{n+1}(P, N_1 \textstyle\coprod N_2) \xrightarrow{\rho\times\sigma} \mathsf{H}^n M \oplus (\mathsf{H}^{n+1}(P, N_1) \oplus \mathsf{H}^{n+1}(P, N_2))\,.$$

• Now one assumes that M is equipped with a framing t. One equips F with the quadratic form q_t and G with the "hyberbolic quadratic form", let us say q_h, defined by

$$\mathsf{q}_\mathsf{h}(z_1, z_2) = \langle\, z_1 \smile z_2 \,,\, [P]\,\rangle$$

(G equipped with this quadratic form is again canonically isomorphic to the quadratic hyperbolic space on $\mathsf{H}^{n+1}(P, N_1)$). One equips E with the quadratic form $\mathsf{q}_t \oplus \mathsf{q}_\mathsf{h}$. The value of this quadratic form on I is given by the following formula:

PROP C. *For all z in $\mathsf{H}^{n+1}(P, N_1 \coprod N_2)$ one has:*

$$\mathsf{q}_t(\rho(z)) + \mathsf{q}_\mathsf{h}(\sigma(z)) = \langle\, \mathsf{v}_{n+1}(\nu_P, t) \smile z \,,\, [P]\,\rangle\,,$$

$\mathsf{v}_{n+1}(\nu_P, t)$ *denoting the relative Wu class defined in* $\mathsf{H}^{n+1}(P, M)$ *using the framing t.*

• Eventually one assumes P equipped with an $(\mathsf{O},\mathsf{fr})^2$-structure.

Then Proposition A and Proposition C imply

$$\mathrm{Arf}(\mathsf{q}_t \oplus \mathsf{q}_\mathsf{h}) = \sum_{i=0}^{n} \langle\, \mathsf{v}_{n+1-i}(\nu_1, t_1) \smile (\mathsf{v}_i(\nu_2, t_2) \smile \mathsf{v}_{n+1}(\nu_2, t_2))\,,\, [P]\,\rangle \quad .$$

Now one has $\mathrm{Arf}(\mathsf{q}_t \oplus \mathsf{q}_\mathsf{h}) = \mathrm{Arf}(\mathsf{q}_t)$ hence one has

$$\mathrm{Arf}(\mathsf{q}_t) = \sum_{i=0}^{n} \langle\, \mathsf{v}_{n+1-i}(\nu_1, t_1) \smile (\mathsf{v}_i(\nu_2, t_2) \smile \mathsf{v}_{n+1}(\nu_2, t_2))\,,\, [P]\,\rangle \,.$$

Q. E. D.