

**Remakes of Browder's paper  
on the Kervaire invariant**

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Edinburgh, april 26th 2011

## 1. Browder's result

Let  $M^{2n}$  ( $n > 0$ ) be a closed manifold equipped with a framing  $t$ . This framing determines a map

$$q_t : H^n M = H^n(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$$

such that one has

$$q_t(x + y) = q_t(x) + q_t(y) + \langle x \smile y, [M] \rangle$$

for all  $x$  and  $y$  in  $H^n M$ . In other words  $q_t$  is a quadratic form (the *Kervaire quadratic form*) associated to the non-degenerate symmetric bilinear form

$$H^n M \times H^n M \rightarrow \mathbb{Z}/2, \quad (x, y) \mapsto \langle x \smile y, [M] \rangle.$$

The *Kervaire invariant* of  $(M, t)$ , let us say  $\kappa(M, t)$ , is the Arf invariant of  $q_t$ . One shows that  $\kappa(M, t)$  depends only on the class of  $(M, t)$  in the cobordism group  $\Omega_{2n}^{\text{fr}}$ .

*Recall of Browder's definition of Kervaire's quadratic form*

Browder considers the map  $Q_t : H^n M \rightarrow \pi_{2n}(S \wedge K_n)$  ( $K_n := K_n(\mathbb{Z}/2, n)$ ) sending a class  $x$  to the class of the composed map

$$S^{2n} \xrightarrow{\text{Thom map}} \text{Th}(\nu_M) \xrightarrow[\sim]{t} S \wedge M_+ \xrightarrow{S \wedge x} S \wedge K_n$$

and defines  $q_t : H^n M \rightarrow \mathbb{Z}/2$  by the composition (the composed map  $\pi_{2n}(S \wedge K_n) \rightarrow \mathbb{Z}/2$  which appears below is in fact an isomorphism)

$$\begin{aligned} H^n M &\xrightarrow{Q_t} \pi_{2n}(S \wedge K_n) = F^1 \pi_{2n}(S \wedge K_n) \\ &\rightarrow \text{Ext}_{\mathbb{A}}^1(H^*(S \wedge K_n), \mathbb{Z}/2) \xrightarrow{\gamma := \text{Sq}^{n+1} \otimes_{\mathbb{A}} \iota_n} \mathbb{Z}/2 \end{aligned}$$

( $\mathbb{A}$  denotes the mod 2 Steenrod algebra, the Adams filtration for the Adams spectral sequence associated to mod 2 cohomology is generically denoted by  $F^0 \supset F^1 \supset F^2 \supset \dots$  and  $\gamma$  is the element in  $\text{Tor}_{\mathbb{A}}^1(\mathbb{Z}/2, H^*(S \wedge K_n))$  corresponding to the unstability relation  $\text{Sq}^{n+1} \iota_n = 0$ ).

**TH 1** (Browder, 1969). *Let  $\chi$  be the canonical anti-automorphism of  $A$ . Then there exists an element  $b_2$  in  $\text{Tor}_2^A(\mathbb{Z}/2, \mathbb{Z}/2)$  represented by a relation in  $A$  of the form*

$$\text{Sq}^{n+1}(\chi \text{Sq}^{n+1}) + \sum_{i=1}^n \text{Sq}^{n+1-i} a_i = 0$$

*such that the Kervaire invariant  $\kappa : \Omega_{2n}^{\text{fr}} \rightarrow \mathbb{Z}/2$  coincides with the following composed map*

$$\Omega_{2n}^{\text{fr}} \cong \pi_{2n} S = F^2 \pi_{2n} S \longrightarrow \text{Ext}_A^2(\mathbb{Z}/2, \mathbb{Z}/2) \xrightarrow{b_2} \mathbb{Z}/2 \quad .$$

**Rm.** A canonical form for the relation defining  $b_2$  is the following one:

$$\sum_{i=0}^n \binom{n+1+i}{i} \text{Sq}^{n+1-i} \chi \text{Sq}^{n+1-i} = 0 \quad .$$

**COR.** *The Kervaire invariant is trivial if  $n + 1$  is not a power of 2. If  $n + 1$  is of the form  $2^k$  then it is non-trivial if and only if in the Adams spectral sequence (the same than the one evoked above) the element  $h_k^2$  in  $E^2$  persists to  $E^\infty$ .*

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Fourty years later we know, thanks to Hill, Hopkins and Ravenel, that  $h_k^2$  does not persist to  $E^\infty$  for  $k \geq 7$  (we knew it persisted for  $k \leq 5$ ).

## 2. Kervaire invariant and manifolds with boundary

**TH 2** (L. 1981). *Let  $b_1$  be the element in  $\text{Tor}_1^{\mathbb{A}}(\mathbb{Z}/2, H^*(\text{MO}/S))$  represented by the following relation in  $H^*\text{MO}$ :*

$$\sum_{i=0}^n \text{Sq}^{n+1-i}(U \smile v_i v_{n+1}) = 0$$

*(observe that  $H^*(\text{MO}/S)$  identifies with the kernel of the restriction  $H^*\text{MO} \rightarrow H^*S = \mathbb{Z}/2$ ). Then the following diagram*

$$\begin{array}{ccc} F^1\pi_{2n+1}(\text{MO}/S) & \longrightarrow & \text{Ext}_{\mathbb{A}}^1(H^*(\text{MO}/S), \mathbb{Z}/2) \\ \downarrow \cong & & \downarrow b_1 \\ \pi_{2n}S & \xrightarrow{\kappa} & \mathbb{Z}/2 \end{array}$$

*is commutative.*

About the relation  $b_1$

- $\chi \text{Sq}^k U := U \smile v_k$

- From the very definition of  $\chi$  one has

$$\sum_{i+j=k} \text{Sq}^i((U \smile v_j) \smile x) = U \smile \text{Sq}^k x$$

for any  $x$  in  $H^*BO$ .

- We take  $k = n + 1$  and  $x = v_{n+1}$  then  $U \smile v_{n+1}^2$  appears on both sides. Hence the relation.

Theorems 1 and 2 are equivalent because  $b_2$  and  $b_1$  correspond to each other *via* the isomorphism

$$\mathrm{Tor}_2^{\mathbb{A}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathrm{Tor}_1^{\mathbb{A}}(\mathbb{Z}/2, H^*(\mathrm{MO}/S)) \quad .$$

*Three of the ingredients coming into the proof of Theorem 2*

- First the interpretation  $\pi_{2n+1}(\mathrm{MO}/S) \cong \Omega_{2n+1}^{\mathrm{O}, \mathrm{fr}}$ .
- Then the two following propositions.

**PROP A.** *Let  $E$  be a finite-dimensional  $\mathbb{Z}/2$ -vector space equipped with a non-degenerate quadratic form  $q$ . Let  $I$  be a subspace of  $E$  with  $I = I^\perp$  and  $u$  be an element in  $E$  such that one has  $q(x) = u \cdot x$  for all  $x$  in  $I$ . Then:*

$$\mathrm{Arf}(q) = q(u) \quad .$$



## *Proof of Proposition A*

*1st case:*  $q(I) = 0$ . Then  $E \approx H(I)$  ( $H(I)$  denotes the hyperbolic quadratic space on  $I$ ),  $u \in I$  and  $q(u) = 0$ .  $\square$

*2nd case:*  $q(I) \neq 0$ . Then  $q|_I$  is a non-zero linear form; let  $J$  be its kernel and  $v$  be an element in  $I$  with  $q(v) = 1$ . Let  $F \subset E$  be the subspace generated by  $u$  and  $v$ . The equality

$$\begin{bmatrix} u.u & u.v \\ v.u & v.v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

implies  $\dim F = 2$  and  $E = F \oplus F^\perp \approx F \oplus H(J)$ .

If  $q(u) = 0$  then  $F \approx H(\mathbb{Z}/2)$ ; if  $q(u) = 1$  then  $F$  is isomorphic to  $(\mathbb{Z}/2)^2$  equipped with the quadratic form  $(x, y) \mapsto x^2 + xy + y^2$ .  $\square$

Now let  $N^{2n+1}$  be a compact manifold whose boundary is  $M$ . Let us take  $E = H^n M$  and  $I = \text{im}(H^n N \rightarrow H^n M)$ . The value of the Kervaire quadratic form on  $I$  is given by the following formula:

**PROP B.** *For all  $y$  in  $H^n N$  one has:*

$$q_t(y|_M) = \langle v_{n+1}(\nu_N, t) \smile y, [N] \rangle \quad ,$$

$v_{n+1}(\nu_N, t)$  denoting the relative Wu class defined in  $H^{n+1}(N, M)$  using the framing  $t$ .

## Proof of Proposition B

The data  $(N, y, M, t)$  determine an element  $[N, y, M, t]$  in  $\pi_{2n+1}((\text{MO}/S) \wedge K_n)$  whose image in  $\pi_{2n}(S \wedge K_n)$  is  $[M, t, y|_M]$ .

One considers the exact sequence of  $A$ -modules

$$0 \rightarrow H^*(\text{MO}/S) \otimes \bar{H}^*K_n \rightarrow H^*\text{MO} \otimes \bar{H}^*K_n \rightarrow \bar{H}^*K_n \rightarrow 0 \quad .$$

- The element  $\iota_n$  in  $\bar{H}^*K_n$  lifts to  $U \otimes \iota_n$  in  $H^*\text{MO} \otimes \bar{H}^*K_n$ ;  $\text{Sq}^{n+1}(U \otimes \iota_n)$  identifies with an element in  $H^*(\text{MO}/S) \otimes \bar{H}^*K_n$ , let us say  $\omega_1$ . The image of  $[N, y, M, t]$  by the homomorphism

$$\pi_{2n+1}((\text{MO}/S) \wedge K_n) \xrightarrow{\omega_1} \mathbb{Z}/2$$

is  $\alpha_t(y|_M)$  .

- Let us denote by  $\omega_2$  the element  $(U \hat{\nu}_{n+1}) \otimes \iota_n$  of  $H^*(\text{MO}/S) \otimes \bar{H}^*K_n$ . The image of  $[N, y, M, t]$  by the homomorphism

$$\pi_{2n+1}((\text{MO}/S) \wedge K_n) \xrightarrow{\omega_2} \mathbb{Z}/2$$

is  $\langle \nu_{n+1}(\nu_N, t) \smile y, [N] \rangle$ .

Now one checks that the images of the classes  $\omega_1$  and  $\omega_2$  in  $\text{Tor}_0^A(\mathbb{Z}/2, H^*(\text{MO}/S) \otimes \bar{H}^*K_n)$  coincide.  $\square$

## *Observations*

- Proposition B implies the invariance of  $\text{Arf}(q_t)$  under framed cobordism.
- Proposition B implies  $v_{n+1}(\nu_N) = 0$ . Therefore the class  $v_{n+1}(\nu_N, t)$  belongs to the image of the connecting homomorphism

$$\partial : H^n M \longrightarrow H^{n+1}(N, M) \quad ;$$

let  $u$  be a class in  $H^n M$  with  $\partial u = v_{n+1}(\nu_N, t)$  then one has

$$q_t(x) = \langle u \smile x, [M] \rangle \quad ,$$

for all  $x$  in  $I = \text{im}(H^n N \rightarrow H^n M)$ . Hence Proposition A gives

$$\text{Arf}(q_t) = q_t(u) \quad .$$

## Proof of Theorem 2

We are left with the task to link the class  $[N, M, t]$  in  $F^1\Omega_{2n+1}^{\text{O,fr}} \cong F^1\pi_{2n+1}(\text{MO}, S)$  with the class  $Q_t(u)$  thought of as a class in  $F^1\pi_{2n+1}(S, S \wedge K_{n+}) = \pi_{2n+1}(S, S \wedge K_{n+}) \cong \pi_{2n}(S \wedge K_n)$ . This is done by thinking of  $N \times [0, 1]$  as a cobordism between  $N$  and  $M \times [0, 1]$ , equipped with some suitable normal structure (the “half-sock cobordism”).

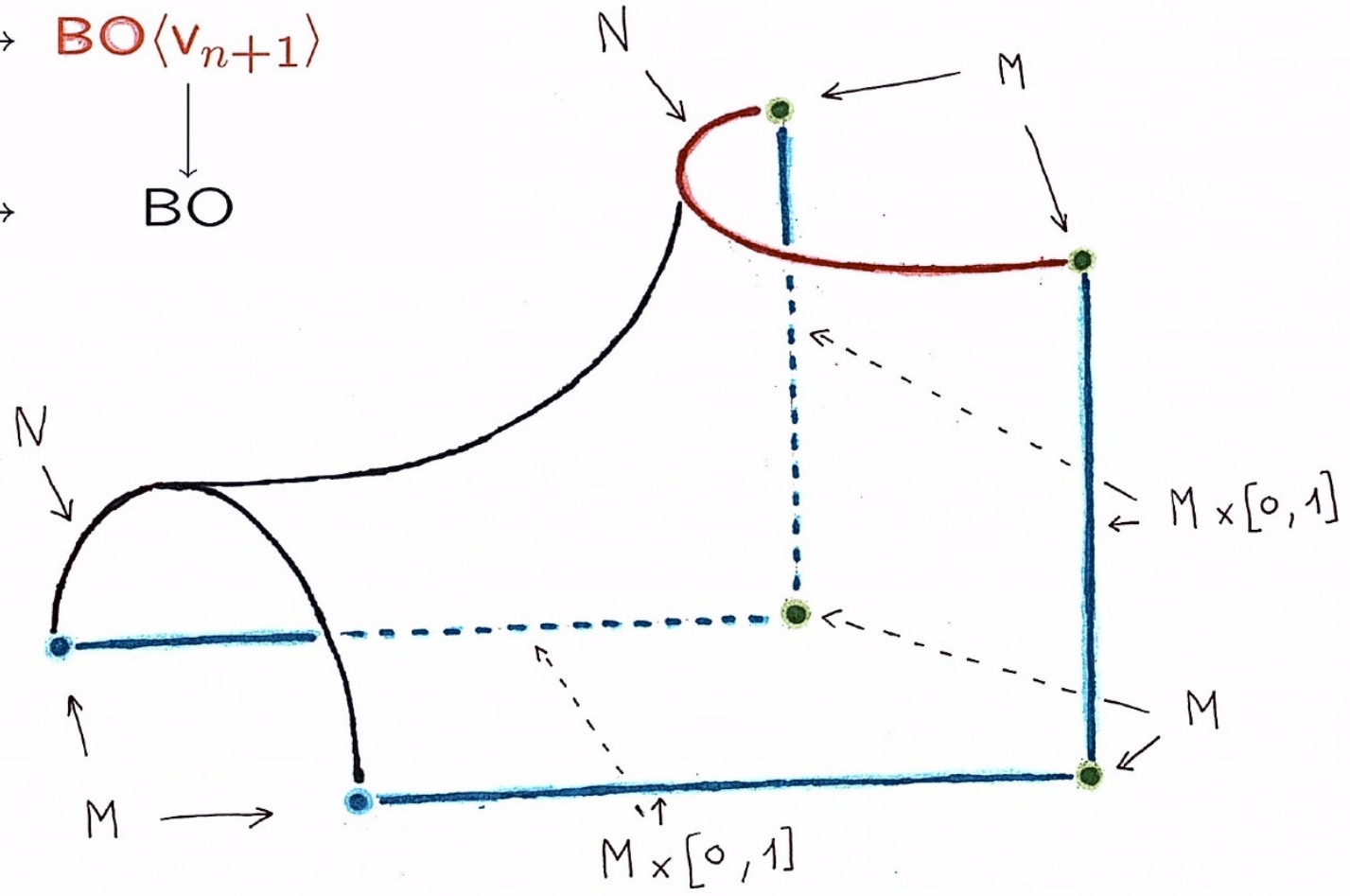
Following Browder, one introduces the space  $\text{BO}\langle v_{n+1} \rangle$  fibre of the map  $v_{n+1} : \text{BO} \rightarrow K_{n+1}$  and one looks at the commutative diagram of spaces

$$\begin{array}{ccc} K_n & \longrightarrow & \text{BO}\langle v_{n+1} \rangle \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{BO} \end{array}$$

and at the induced diagram of stable vector bundles.

# The half-sock cobordism

$$\begin{array}{ccc}
 \mathbf{K}_n & \rightarrow & \mathbf{BO}\langle v_{n+1} \rangle \\
 \downarrow & & \downarrow \\
 * & \rightarrow & \mathbf{BO}
 \end{array}$$



The half-sock cobordism shows that the images of  $[N, M, t]$  and  $Q_t(u)$  in  $F^1\pi_{2n+1}(\text{MO}, \text{MO}\langle v_{n+1} \rangle)$  (diagram below) coincide.

$$\begin{array}{ccccc}
 F^1\pi_{2n+1}(\text{MO}, S) & \longrightarrow & F^1\pi_{2n+1}(\text{MO}, \text{MO}\langle v_{n+1} \rangle) & \longleftarrow & F^1\pi_{2n+1}(S, S \wedge K_{n+}) \\
 \downarrow b_1 & & \downarrow \beta & & \downarrow \Sigma\gamma \cong \\
 \mathbb{Z}/2 & \quad \quad \quad = & \mathbb{Z}/2 & \quad \quad \quad = & \mathbb{Z}/2
 \end{array}$$

Now let  $\beta \in \text{Tor}_1^A(\mathbb{Z}/2, H^*(\text{MO}, \text{MO}\langle v_{n+1} \rangle))$  be the element corresponding to the relation

$$\sum_{i=0}^n \text{Sq}^{n+1-i} (U \smile v_i \widehat{V}_{n+1}) = 0$$

in  $H^*(\text{MO}, \text{MO}\langle v_{n+1} \rangle)$ , then  $\beta$  induces  $b_1$  in  $\text{Tor}_1^A(\mathbb{Z}/2, H^*(\text{MO}, S))$  and  $\Sigma\gamma = \text{Sq}^{n+1} \otimes_A \Sigma\iota_n$  in  $\text{Tor}_1^A(\mathbb{Z}/2, H^*(S, S \wedge K_{n+}) = \Sigma H^*(S \wedge K_n))$ .

Eventually we do get

$$\langle b_1, [N, M, t] \rangle = q_t(u) = \kappa(M, t) \quad .$$

Q. E. D.

### 3. Kervaire invariant and manifolds with a corner of codimension 2

Now we climb further up in the MO-spectral sequence:

**TH 3** (Miller-L.). *Let  $b$  be the element in  $H^*(MO/S \wedge MO/S)$  whose image in  $H^*(MO \wedge MO)$  is:*

$$\sum_{i=0}^n (U \smile v_{n+1-i}) \times (U \smile v_i v_{n+1}) \quad .$$

*Then the following diagram*

$$\begin{array}{ccc} \pi_{2n+2}(MO/S \wedge MO/S) & \longrightarrow & H_{2n+2}(MO/S \wedge MO/S) \\ \downarrow & & \downarrow b \\ \pi_{2n}S & \xrightarrow{\kappa} & \mathbb{Z}/2 \end{array}$$

*is commutative.*



Let  $b_0$  be the class of  $b$  in  $\text{Tor}_0^{\mathbb{A}}(\mathbb{Z}/2, H^*(\text{MO}/S \wedge \text{MO}/S))$ .

Theorem 1 (Browder's theorem) and Theorem 3 are equivalent because the left vertical arrow of the diagram above is surjective and  $b_0$  is the image of  $b_2$  under the monomorphism

$$\text{Tor}_2^{\mathbb{A}}(\mathbb{Z}/2, \mathbb{Z}/2) \hookrightarrow \text{Tor}_0^{\mathbb{A}}(\mathbb{Z}/2, H^*(\text{MO}/S \wedge \text{MO}/S)) \quad .$$

### *Proof of Theorem 3*

- $\pi_{2n+2}(\text{MO}/S \wedge \text{MO}/S) \cong \Omega_{2n+2}^{(\text{O},\text{fr})^2}$ .

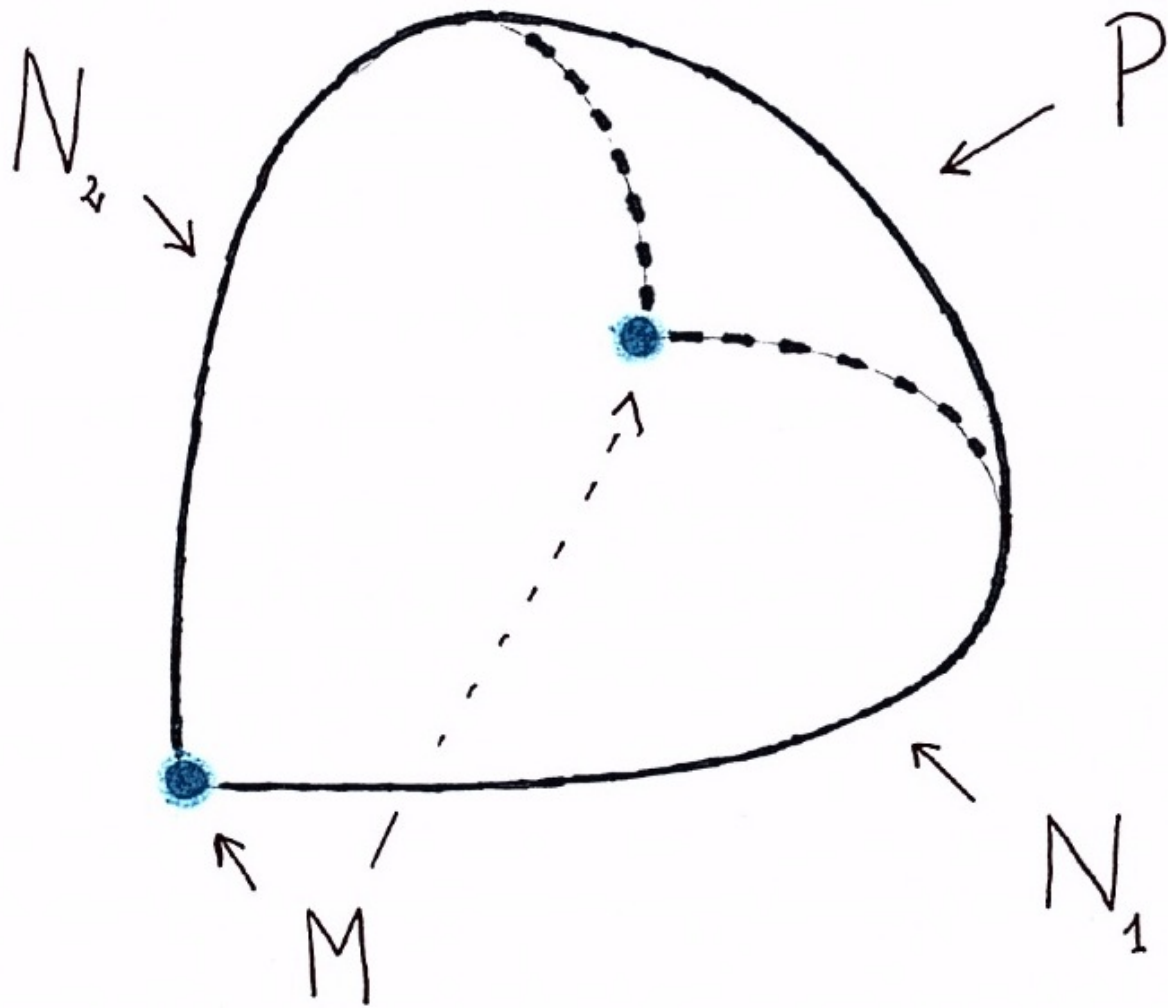
Let  $P$  be a  $(2n+2)$ -dimensional compact manifold with a corner of codimension 2. One denotes by  $M$  this corner and by  $N$  the boundary of  $P$ ,  $N$  is the union of two manifolds with boundary,  $N_1$  and  $N_2$ , whose intersection is  $M$  (their common boundary).

One assumes one has a splitting  $\nu_P = \nu_1 \oplus \nu_2$  such that:

- there exists a trivialisation, let us say  $t_1$ , of the restriction of  $\nu_1$  to  $N_1$ ;
- there exists a trivialisation, let us say  $t_2$ , of the restriction of  $\nu_2$  to  $N_2$ .

$t_1$  and  $t_2$  determine a trivialisation of  $\nu_M$  which is denoted by  $t$ .

The group  $\Omega_{2n+2}^{(\text{O},\text{fr})^2}$  is the cobordism group of such  $P$ s.



- Given a  $(2n+2)$ -dimensional compact manifold  $P$  with a corner of codimension 2, one produces a  $\mathbb{Z}/2$ -vector space  $E$  equipped with a non-degenerate symmetric bilinear form and a lagrangian subspace  $I$ . In this part there are no normal structures involved, we are just playing with Poincaré duality (that is to say the conjunction of Spanier-Whitehead duality and Thom isomorphism).

One sets  $F = H^n M$ ;  $F$  is equipped with the “intersection pairing”.

One sets  $G = H^{n+1}(P, N_1) \oplus H^{n+1}(P, N_2)$ ; one equips  $G$  with the symmetric bilinear form

$$(z_1, z_2) \cdot (z'_1, z'_2) = \langle z_1 \smile z'_2 + z'_1 \smile z_2, [P] \rangle$$

( $G$  is canonically isomorphic to the hyperbolic space on  $H^{n+1}(P, N_1)$  because  $H^{n+1}(P, N_2)$  is Poincaré-dual of  $H_{n+1}(P, N_1)$ ).

One defines  $E$  as the orthogonal sum  $F \oplus G$ . One defines  $I$  as the image of the natural map

$$H^{n+1}(P, N_1 \amalg N_2) \xrightarrow{\rho \times \sigma} H^n M \oplus (H^{n+1}(P, N_1) \oplus H^{n+1}(P, N_2)).$$

- Now one assumes that  $M$  is equipped with a framing  $t$ . One equips  $F$  with the quadratic form  $q_t$  and  $G$  with the “hyberbolic quadratic form”, let us say  $q_h$ , defined by

$$q_h(z_1, z_2) = \langle z_1 \smile z_2, [P] \rangle$$

( $G$  equipped with this quadratic form is again canonically isomorphic to the quadratic hyperbolic space on  $H^{n+1}(P, N_1)$ ). One equips  $E$  with the quadratic form  $q_t \oplus q_h$ . The value of this quadratic form on  $I$  is given by the following formula:

**PROP C.** For all  $z$  in  $H^{n+1}(P, N_1 \amalg N_2)$  one has:

$$q_t(\rho(z)) + q_h(\sigma(z)) = \langle v_{n+1}(\nu_P, t) \smile z, [P] \rangle,$$

$v_{n+1}(\nu_P, t)$  denoting the relative Wu class defined in  $H^{n+1}(P, M)$  using the framing  $t$ .

- Eventually one assumes  $P$  equipped with an  $(O, \text{fr})^2$ -structure.

Then Proposition A and Proposition C imply

$$\text{Arf}(\mathfrak{q}_t \oplus \mathfrak{q}_h) = \sum_{i=0}^n \langle v_{n+1-i}(\nu_1, t_1) \smile (v_i(\nu_2, t_2) \smile v_{n+1}(\nu_2, t_2)), [P] \rangle .$$

Now one has  $\text{Arf}(\mathfrak{q}_t \oplus \mathfrak{q}_h) = \text{Arf}(\mathfrak{q}_t)$  hence one has

$$\text{Arf}(\mathfrak{q}_t) = \sum_{i=0}^n \langle v_{n+1-i}(\nu_1, t_1) \smile (v_i(\nu_2, t_2) \smile v_{n+1}(\nu_2, t_2)), [P] \rangle .$$

Q. E. D.