Geometric approach to stable homotopy groups of spheres: The Kervaire invariant

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The Kervaire invariant and stable homotopy theory, Edinburgh
27 April 2011

Let

$$f : M^{n-1} \looparrowright \mathbb{R}^n, \quad n = 4k+2, \quad n \geq 2$$

be a smooth generic immersion of a closed manifold of codimension 1,

$$g : N^{n-2} \looparrowright \mathbb{R}^n$$

be the immersion of the double points intersection of g.

The **Kervaire invariant** of f is defined by

$$\Theta^{sf}(f) = \langle \eta_N^{\frac{n-2}{2}}; [N^{n-2}] \rangle,$$

where $\eta_N = w_2(N^{n-2})$ is the second normal Stiefel-Whitney class of N^{n-2}.

In particular, if $n = 2$, $\Theta_{sf}(f)$ is the parity of the number of self-intersection points of the regular curve f on the plane $\mathbb{R}^2$.

Let us denote by $Imm^{sf}(n-1, 1)$ the cobordism group of immersions in the codimension 1 of (non-oriented) closed $(n-1)$–manifold ("sf" stands for skew-framed).

Theorem

The Kervaire invariant is a well-defined homomorphism:

$$\Theta_{sf} : Imm^{sf}(4k+1,1) \longrightarrow \mathbb{Z}/2.$$

1. This homomorphism is trivial if $4k+2 \neq 2^l - 2$, $l \geq 2$.
2. For $k = 0,1,3,7,15$ the homomorphism Θ^{sf} is an epimorphism.

Part 1 is the Theorem by W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math., (2) 90 (1969) 157-186. A special case of this theorem was discovered by M. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv., 34 (1960) 257-270.

Part 2 (in the case $k = 7$) was proved by M.E. Mahowald and M.C. Tangora, *Some differentials in the Adams spectral sequence*, Topology 6 (1967), 349-369. The case $k = 15$ was proved by M. G. Barratt, J. D. S. Jones and M. E. Mahowald, *Relations amongst Toda brackets and the Kervaire invariant in dimension* 62, J.London Math.Soc. (2) 30 (1984), no. 3, 533-550.

Main Theorem

There exists an integer l_0, such that for an arbitrary $l \geq l_0$, the Kervaire invariant

$$\Theta^{sf} : Imm^{sf}(2^l - 3, 1) \longrightarrow \mathbb{Z}/2$$

is the trivial homomorphism.

The proof of this theorem is based on the approach proposed by P.J.Eccles *Codimension One Immersions and the Kervaire Invariant One Problem*, Math. Proc. Cambridge Phil. Soc., vol.90 (1981) 483-493.

Remark

The Main Theorem is a particular case of the Hill-Hopkins-Ravenel theorem ($l_0 = 8$).

A commutative diagram that we use to define dihedral structure of self-intersection manifold of skew-framed immersions:

$$\begin{array}{ccc} Imm^{sf}(n-1,1) & \stackrel{J^{sf}_k}{\longrightarrow} & Imm^{sf}(n-k,k) \\ \downarrow \delta^{[2]} & & \downarrow \delta^{[2]}_k \\ Imm^{\mathbb{Z}/2^{[2]}}(n-2,2) & \stackrel{J^{[2]}_k}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k). \end{array}$$

A commutative diagram that we use to define the Kervaire invariant in the codimension k:

$$\begin{array}{cccc} Imm^{sf}(n-k,k) & \stackrel{\Theta^{sf}_k}{\longrightarrow} & \mathbb{Z}/2 & \frac{skew-framed}{immersions} \\ \downarrow \delta^{[2]}_k & & \Vert & \\ Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k) & \stackrel{\Theta^{[2]}_k}{\longrightarrow} & \mathbb{Z}/2 & \frac{dihedral-framed}{immersions}. \end{array}$$

Structural groups of immersions

Let us consider the following collection of $(d-1)$ sets

$$\Upsilon_d, \Upsilon_{d-1}, \dots, \Upsilon_2,$$

where each set consists of proper coordinate subspaces of $\mathbb{R}^{2^{d-1}}$.

The set of the subspaces

$$\Upsilon_i, \quad 2 \le i \le d,$$

(we will use only the case $d \le 5$) consists of 2^{i-1} coordinate subspaces generated by the basis vectors:

$$((\mathbf{e}_1, \dots \mathbf{e}_{2^{d-i}}), \dots, (\mathbf{e}_{2^{d-1}-2^{d-i}+1}, \dots, \mathbf{e}_{2^{d-1}})).$$

Let us denote by $\mathbb{Z}/2^{[d]}$ the subgroup of the group

$$\mathbb{Z}/2 \wr \Sigma_{2^{d-1}} \subset O(2^{d-1})$$

under the following condition:

– the transformation

$$\mathbb{Z}/2^{[d]} \times \mathbb{R}^{2^{d-1}} \to \mathbb{R}^{2^{d-1}}$$

admits the invariant collection of sets

$$\Upsilon_d, \Upsilon_{d-1}, \dots, \Upsilon_2.$$

In particular, in the case $d = 2$ we get that Υ_2 contains only one collection of subspaces and this collection is $((\mathbf{e}_1), (\mathbf{e}_2))$. Therefore $\mathbb{Z}/2^{[2]}$ is the dihedral group.

$$
\begin{array}{ccc}
Imm^{\mathbb{Z}/2^{[2]}}(n-2,2) & \longrightarrow & Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k) \\
\downarrow \delta^{[3]} & & \downarrow \delta_k^{[3]} \\
Imm^{\mathbb{Z}/2^{[3]}}(n-4,4) & \stackrel{J_k^{[3]}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k) \\
\downarrow \delta^{[4]} & & \downarrow \delta_k^{[4]} \\
Imm^{\mathbb{Z}/2^{[4]}}(n-8,8) & \stackrel{J_k^{[4]}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k) \\
\downarrow \delta^{[5]} & & \downarrow \delta_k^{[5]} \\
Imm^{\mathbb{Z}/2^{[5]}}(n-16,16) & \stackrel{J_k^{[5]}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[5]}}(n-16k,16k)
\end{array}
$$

$$
\begin{array}{ccc}
Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k) & \longrightarrow & \mathbb{Z}/2 \\
\downarrow \delta_k^{[3]} & & \| \\
Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k) & \stackrel{\Theta_k^{[3]}}{\longrightarrow} & \mathbb{Z}/2 \\
\downarrow \delta_k^{[4]} & & \| \\
Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k) & \stackrel{\Theta_k^{[4]}}{\longrightarrow} & \mathbb{Z}/2 \\
\downarrow \delta_k^{[5]} & & \| \\
Imm^{\mathbb{Z}/2^{[5]}}(n-16k,16k) & \stackrel{\Theta_k^{[5]}}{\longrightarrow} & \mathbb{Z}/2
\end{array}
$$

$$
\begin{array}{cccc}
\mathbf{I}_b \times \dot{\mathbf{I}}_b & \subset & \mathbb{Z}/2^{[2]} & \frac{\textit{abelian}}{\textit{structure}} \\
\downarrow & & i^{[3]} \downarrow & \\
\mathbf{I}_a \times \dot{\mathbf{I}}_b & \subset & \mathbb{Z}/2^{[3]} & \frac{\textit{bicyclic}}{\textit{structure}} \\
\downarrow & & i^{[4]} \downarrow & \\
\mathbf{I}_a \times \dot{\mathbf{I}}_a & \subset & \mathbb{Z}/2^{[4]} & \frac{\textit{bicyclic}}{\textit{structure}} \\
\downarrow & & i^{[5]} \downarrow & \\
\mathbf{Q} \times \mathbb{Z}/4 & \subset & \mathbb{Z}/2^{[5]} & \frac{\textit{quaternionic}}{\textit{structure}}
\end{array}
$$

Dihedral group

The dihedral group (of the order 8) $\mathbb{Z}/2^{[2]} \subset O(2)$:

$$\{a, b \quad | \quad a^4 = b^2 = e, [a, b] = a^2\}.$$

Let $\{\mathbf{f}_1, \mathbf{f}_2\}$ be the standard base of the plane $\mathbb{R}^2$. The element a is represented by the rotation through the angle $\frac{\pi}{2}$:

$$f_1 \mapsto f_2; \quad f_2 \mapsto -f_1.$$

The element b is represented by the permutation of the base vectors

$$f_1 \mapsto f_2; \quad f_2 \mapsto f_1.$$

Elementary 2-group

The elementary subgroup $\mathbf{I}_b \times \dot{\mathbf{I}}_b \subset \mathbb{Z}/2^{[2]}$ of the rank 2:

$$\{a^2, b \quad | \quad a^4 = b^2 = e, [a^2, b] = e\}.$$

This group preserves the vectors $\mathbf{f}_1 + \mathbf{f}_2$, $\mathbf{f}_1 - \mathbf{f}_2$.

The group $(\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}$

Define the automorphism

$$\chi^{[2]} : \mathbf{I}_b \times \dot{\mathbf{I}}_b \to \mathbf{I}_b \times \dot{\mathbf{I}}_b$$

by the external conjugation of the subgroup $\mathbf{I}_b \times \dot{\mathbf{I}}_b \subset \mathbb{Z}/2^{[2]}$ by the element $ab \in \mathbb{Z}/2^{[2]}$.

Define the group

$$(\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \qquad (1)$$

as the factorgroup of the group $(\mathbf{I}_b \times \dot{\mathbf{I}}_b) * \mathbb{Z}$ by the relation $zxz^{-1} = \chi^{[2]}(x)$, where $z \in \mathbb{Z}$ is the standard generator, $x \in \mathbf{I}_b \times \dot{\mathbf{I}}_b$ is an arbitrary element. The group (1) is a particular example of a semi-direct product of groups $A \rtimes_\phi B$, $A = \mathbf{I}_b \times \dot{\mathbf{I}}_b$, $B = \mathbb{Z}$, where $\phi : B \to \mathrm{Aut}(A)$.

The classifying Eilenberg-Mac Lane space $K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1)$

Take a skew-product of the standard circle S^1 and the space $K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1)$, the shift $K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \to K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1)$, of the cyclic covering is given by the automorphism $\chi^{[2]}$. The standard inclusion

$$K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \subset K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1).$$

is well defined.

Homology with local coefficients

Define the group

$$H_i(K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]).$$

Consider the group ring $\mathbb{Z}/2[\mathbb{Z}/2] = \{a + bt\}$, $a, b \in \mathbb{Z}/2$, $t \in \mathbb{Z}/2$. Consider the following local system of the coefficients, associated with the homomorphism $\rho : (\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \to Aut(\mathbb{Z}[\mathbb{Z}/2])$. Assume $x \in (\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}$. Define the automorphism $\rho(x)$ of $\mathbb{Z}/2[\mathbb{Z}/2]$ by the multiplication on the generator t, if $p_{b \times \dot{b}}(x) \equiv 1 \pmod 2$ and by the identity automorphism, if $p_{b \times \dot{b}}(x) \equiv 0 \pmod 2$, where

$$p_{b \times \dot{b}} : (\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z}.$$

Splitting of homology

Define the subgroup

$$H_i(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1); \mathbb{Z}/2) \cong D_i(\mathbf{I}_b \times \dot{\mathbf{I}}_b; \mathbb{Z}/2[\mathbb{Z}/2]) \subset$$

$$H_i(K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])$$

by the following formula (is induced by the standard inclusion):

$$\mathrm{Im}(H_i(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1); \mathbb{Z}/2[\mathbb{Z}/2]) \to H_i(K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]).$$

Define the splitting

$$\Delta^{[2]} : H_i(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1); \mathbb{Z}/2[\mathbb{Z}/2]) \to D_i(\mathbf{I}_b \times \dot{\mathbf{I}}_b; \mathbb{Z}/2[\mathbb{Z}/2])$$

by the formula $\Delta^{[2]}(X + Yt) = X + \chi^{[2]}_*(Y)$.

Reduction of the dihedral framing

Define the epimorphism

$$\Phi^{[2]} : (\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z}/2^{[2]}$$

by the formula: $\Phi^{[2]}(z) = ab$ (the element ab corresponds to the reflection of the first vector of the standard base), the restriction

$$\Phi^{[2]}|_{\mathbf{I}_b \times \dot{\mathbf{I}}_b \times \{0\}} : \mathbf{I}_b \times \dot{\mathbf{I}}_b \subset \mathbb{Z}/2^{[2]}$$

is the standard inclusion.

Reduction of the characteristic class

Define

$$(\Phi^{[2]})^*(\tau_{[2]}) = \tau_{b \times \dot{b}},$$

where $\tau_{b \times \dot{b}} \in H^2(K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1))$, $\tau_{[2]} \in H^2(K(\mathbb{Z}^{[2]}, 1)$.

Definition of abelian structure

$$x = [(f, \Xi, \kappa_M)] \in Imm^{sf}(n-k, k),$$

–$f : M^{n-k} \looparrowright \mathbb{R}^n$,

–κ_M is a line bundle over M^{n-k},

–Ξ is a skew-framing of the normal bundle of f, i.e. an isomorphism $\Xi : \nu_f = k\kappa_M$.

$$y = \delta_k^{[2]}(x) = [(g, \Psi, \eta_N)] \in Imm^{\mathbb{Z}^{[2]}}(n-2k, 2k),$$

–$g : N^{n-2k} \looparrowright \mathbb{R}^n$,

–η_N is a $\mathbb{Z}/2^{[2]}$–bundle over N^{n-2k},

–Ψ is a dihedral framing of the normal bundle of f, i.e. an isomorphism $\Psi : \nu_g = k\eta_N$.

We say that (f, Ξ, κ_M) admits an abelian structure if the cobordism class $y = [(g, \Psi, \eta_N)] = \delta_k^{[2]}(x)$ satisfies the following conditions:

1. There exists a mapping

$$\eta_{b \times \dot{b}} : N^{n-2k} \to K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1).$$

2. The restrictions of the mapping $\eta_{b \times \dot{b}, N}$ on the submaniflod

$$N_{\eta^{7k}}^{n-16k} \subset N^{n-2k}, \qquad (2)$$

which represents the (homology) Euler class $[\eta_N^{7k}]^{op} \in H_{n-16k}(N^{n-2k})$ of the bundle $7k\eta_N$, is given by the composition:

$$\eta_{b \times \dot{b}} : N_{\eta^{7k}}^{n-16k} \to K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \to K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1).$$

3. The restriction of the characteristic mapping η_N on the submanifold (2) is given by the composition $\Phi^{[2]} \circ \eta_{b \times \dot{b}, N}$:

$$N_{\eta^{7k}}^{n-16k} \to K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \to K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbb{Z}/2^{[2]}, 1).$$

Example 1

Let a skew-framed immersion (f, Ξ, κ_M), $f : M^{n-k} \looparrowright \mathbb{R}^n$ be represented an element $x \in Imm^{sf}(n-k, k)$, $n > 16k$ and be a $\mathbf{I}_b \times \dot{\mathbf{I}}_b$-framed immersion. Then the skew-framed immersion (f, Ξ, κ_M) admits an abelian structure.

Example 2

Let a skew-framed immersion (f, Ξ, κ_M), $f : M^{n-k} \looparrowright \mathbb{R}^n$ be represented an element $x \in Imm^{sf}(n-k, k)$, $n > 16k$ and be a $(\mathbf{I}_b \times \dot{\mathbf{I}}_b)_{\chi^{[2]}} \int \mathbb{Z}$-framed immersion, and the restriction of this immersion on the submaniflod

$$N^{n-16k}_{\eta^{7k}} \subset N^{n-2k}$$

is a $\mathbf{I}_b \times \dot{\mathbf{I}}_b$-framed immersion. Then the skew-framed immersion (f, Ξ, κ_M) admits an abelian structure.

Define the codimension 2 submaniflod

$$N^{n-2k-2}_{\circ} \subset N^{n-2k},$$

which represents the Euler class $\eta^*_{b \times \dot{b}}(\tau_{b \times \dot{b}})$. Define the codimension 16 submaniflod

$$N^{n-16k-16}_{\eta^{7k}\circ} \subset N^{n-16k}_{\eta^{7k}},$$

which represents the Euler class $\eta^*_{b \times \dot{b}}(\tau^4_{b \times \dot{b}})$. Define the element

$$\eta^{loc}_{b \times \dot{b},*}([N^{n-16k-16}_{\eta^{7k}\circ}]) \in D_{n-16k-16}(\mathbf{I}_b \times \dot{\mathbf{I}}_b; \mathbb{Z}/2[\mathbb{Z}/2]). \qquad (3)$$

Lemma-Definition

Let a skew-framed immersion (f, Ξ, κ_M) be admitted an abelian structure. Then for the element $y = \delta^{[2]}_k(x) = [(g, \Psi, \eta_N)] \in Imm^{\mathbb{Z}^{[2]}}(n-2k, 2k)$, The decomposition of the image of the homology class (3) over the standard basis of the group

$$D_{n-16k-16}(\mathbf{I}_b \times \dot{\mathbf{I}}_b; \mathbb{Z}/2[\mathbb{Z}/2]) = H_{n-16k-16}(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1))$$

contains not more then the only non-trivial element, which is determined by the coefficient of the generator $t_{b,j} \otimes t_{\dot{b},j}$, $j = \frac{n-16k-16}{2}$. This coefficient coincides with the Kervaire invariant for the $\mathbb{Z}/2^{[2]}$–framed immersion (g, Ψ, η_N).

Compression

A skew-framed cobordism class

$$x = [(f, \Xi, \kappa_M)] \in Imm^{sf}(n-k, k)$$

admits a compression of order q, if this class is represented by a triple, such that

$$\kappa_M = I \circ \kappa_{(q)},$$

$\kappa_{(q)} : M^{n-k} \to \mathbb{R}\mathrm{P}^{n-k-q-1}$, $I : \mathbb{R}\mathrm{P}^{n-k-q-1} \subset \mathbb{R}\mathrm{P}^{\infty}$.

Compression theorem

For an arbitrary q there exists an integer $l_0 = l_0(q)$, such that an arbitrary element $x \in Imm^{sf}(2^l - 3, 1)$, $l \ge l_0$, admits a compression of order q.

Abelian structure theorem

Let $2^l - 16k = n - 16k = 2^{\sigma} - 2$, $k \equiv 0 \pmod 2$, $(l >> \sigma \ge 5)$. Assume that the element $x \in Imm^{sf}(n-k, k)$ admits a compression of order $q = 2^{\sigma-1} - 2$. Then the element x be represented by a triple (f, Ξ, κ_M), which admits an abelian structure.

Cyclic group

The cyclic index 2 subgroup of order 4:

$$\mathbf{I}_a = \{a \quad | \quad a^4 = e\} \subset \mathbb{Z}/2^{[2]}.$$

Bicyclic subgroups

The bicyclic index 2^{11} subgroup of the order 16:

$$\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbb{Z}/2^{[4]}.$$

Denote the generators by

$$t_{a,j}, \in H_j(K(\mathbf{I}_a, 1)); \quad t_{\dot{a},j} \in H_j(K(\dot{\mathbf{I}}_a, 1)), \qquad j \equiv 1 \pmod 2.$$

Define automorphism of order 2:

$$\chi^{[4]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbf{I}_a \times \dot{\mathbf{I}}_a.$$

Define the group

$$(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}.$$

Define the natural homomorphism

$$\Phi^{[4]} : (\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z} \to \mathbb{Z}/2^{[4]}.$$

The classifying space $K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)$ is the semi-direct product of the standard circle S^1 and the space $K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1)$.

Homology with local coefficients

Define the group

$$H_i(K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]).$$

Define the subgroup

$$H_i(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2) \cong D_i(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) \subset$$
$$H_i(K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]).$$

Define the splitting

$$\Delta^{[4]} : H_i(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2[\mathbb{Z}/2]) \to D_i(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]).$$

Structural groups of framings

$$\begin{array}{ccc}
(\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} & \stackrel{\Phi^{[2]}}{\longrightarrow} & \mathbb{Z}/2^{[2]} \\
i_{b\times \dot{b}, \mathbf{I}_a \times \dot{\mathbf{I}}_b} \downarrow & & i^{[3]} \downarrow \\
(\mathbf{I}_a \times \dot{\mathbf{I}}_b) \int_{\chi^{[3]}} \mathbb{Z} & \stackrel{\Phi^{[3]}}{\longrightarrow} & \mathbb{Z}/2^{[3]} \\
i_{\mathbf{I}_a \times \dot{\mathbf{I}}_b, \mathbf{I}_a \times \dot{\mathbf{I}}_a} \downarrow & & i^{[4]} \downarrow \\
(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]}}{\longrightarrow} & \mathbb{Z}/2^{[4]}.
\end{array}$$

$$x \in Imm^{sf}(n-k,k),$$

$$y = \delta^{[2]}(x) \in Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k),$$

$$z = \delta^{[3]} \circ \delta^{[2]} \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k),$$

$$u = \delta^{[4]} \circ \delta^{[3]} \circ \delta^{[2]}(x),$$

$$u = [(h,\Lambda,\zeta_L)] \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k).$$

–$h : L^{n-8k} \looparrowright \mathbb{R}^n$,

–ζ_L is a $\mathbb{Z}/2^{[4]}$–bundle over L^{n-8k},

–Λ is a an 8–dimensional $\mathbb{Z}/4^{[4]}$-framing of the normal bundle of h, i.e. an isomorphism $\Lambda : \nu_h \simeq k\zeta_L$.

Define the codimension 8 submaniflod

$$L_{\circ}^{n-8k-8} \subset L^{n-8k},$$

which represents the Euler class $\zeta^*_{a\times\dot{a}}(\tau'_{a\times\dot{a}})$,

$$\tau'_{a\times\dot{a}} \in H^8(K((\mathbf{I}_a \times \dot{\mathbf{I}}_a)\int_{\chi^{[4]}} \mathbb{Z},1)),$$

$$\tau'_{a\times\dot{a}} = \Phi^{[4]*}(\tau_{[4]}), \quad \tau_{[4]} \in H^8(K(\mathbb{Z}/2^{[4]},1))$$

Define the codimension 16 submaniflod

$$L_{\zeta^k \circ}^{n-16k-16} \subset L_{\zeta^k}^{n-16k},$$

which represents the Euler class $\zeta^*_{a\times\dot{a}}(\tau'^2_{a\times\dot{a}})$. Define the element

$$\zeta^{loc}_{a\times\dot{a},*}([L_{\zeta^k \circ}^{n-16k-16}]) \in D_{n-16k-16}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]). \qquad (4)$$

Definition of bicyclic structure

We say that a triple (g, Ψ, η_N), which represents an element in $Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$, admits a bicyclic structure if for $\delta^{[4]}(z) = [(h, \Lambda, \zeta_L)]$ there exists a mapping

$$\zeta_{a \times \dot{a}} : L^{n-8k} \to K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1),$$

for which the decomposition of the image of the homology class $\zeta^{loc}_{a \times \dot{a},*}([L^{n-16k-16}_{\zeta^k \circ}])$ over the standard basis of the group

$$D_{n-16k-16}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) = H_{n-16k-16}(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1))$$

contains not more then the only non-trivial element, which is determined by the coefficient of the generator $t_{a,j} \otimes t_{\dot{a},j}$, $j = \frac{n-16k-16}{2}$. This coefficient coincides with the Kervaire invariant for the $\mathbb{Z}/2^{[4]}$–framed immersion (h, Λ, ζ_L).

Bicyclic structure theorem

Assume that the dimensional restriction of Abelian structure theorem is satisfied and the element $x \in Imm^{sf}(n-k, k)$ admits a compression of order $q = 2^{\sigma-1} - 2$. Then the cobordism class $z = \delta^{[3]} \circ \delta^{[2]} \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$ be represented by a triple (g, Ψ, η_N), which admits a bicyclic structure.

Quaternionic groups

The quaternionic group of the order 8:

$$\mathbf{Q} = \{\mathbf{i}, \mathbf{j}, \mathbf{k} \quad | \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik},$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\}.$$

Define the subgroups

$$i_{\mathbf{Q} \times \mathbb{Z}/4} : \mathbf{Q} \times \mathbb{Z}/4 \subset \mathbb{Z}/2^{[5]},$$

$$i_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2 \subset \mathbb{Z}/2^{[5]}.$$

Define the involutions

$$\chi^{[5]} : \mathbf{Q} \times \mathbb{Z}/4 \to \mathbf{Q} \times \mathbb{Z}/4,$$

$$\chi^{[5]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2 \to \mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2.$$

Define groups

$$(\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z},$$

$$(\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z}$$

Define the following homomorphisms:

$$\Phi^{[5]} : (\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/2^{[5]},$$

$$\Phi^{[5]} : (\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/2^{[5]}.$$

Commutative diagrams are well defined:

$$\begin{array}{ccc} (\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]} \tilde{\times} \Phi^{[4]}}{\longrightarrow} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\ i_{\mathbf{Q}\times\mathbb{Z}/4} \downarrow & & i_{[5]} \downarrow \\ (\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z} & \stackrel{\Phi^{[5]}}{\longrightarrow} & \mathbb{Z}/2^{[5]}. \end{array}$$

$$\begin{array}{ccc} (\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]} \tilde{\times} \Phi^{[4]}}{\longrightarrow} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\ i_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2} \downarrow & & i_{[5]} \downarrow \\ (\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z} & \stackrel{\Phi^{[5]}}{\longrightarrow} & \mathbb{Z}/2^{[5]}. \end{array}$$

Definition of quaternionic structure

Let a $\mathbb{Z}/2^{[4]}$–framed immersion (g, Ψ, η_N), $g : N^{n-8k} \looparrowright \mathbb{R}^n$ be represented an element $v \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k)$. We say that v admits a quaternionic structure if there exists a mapping

$$\eta_{\mathbf{I}_a \times \dot{\mathbf{I}}_a, N} : N^{n-8k} \to K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1),$$

which satisfies conditions of a bicyclic structure and the following conditions:

Let a $\mathbb{Z}/2^{[5]}$–framed immersion (h, Λ, ζ_L), $h : L^{n-16k} \looparrowright \mathbb{R}^n$, be the immersion of self-intersection points of g and be represented an element $w = \delta^{[5]}(v) \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)$. Let the manifold L^{n-16k} be represented by the following disjoint union of two components:

$$L^{n-16k} = L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} \cup L^{n-16k}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2},$$

and the following diagrams of canonical 2-sheeted coverings are commutative:

$$\begin{array}{ccc} \eta_{\mathbf{I}_a \times \dot{\mathbf{I}}_a} : \bar{L}^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} & \longrightarrow & K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1) \\ \downarrow & & \downarrow \\ \zeta_{\mathbf{Q}\times\mathbb{Z}/4} : L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} & \longrightarrow & K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, 1), \end{array}$$

$$\begin{array}{ccc} \eta_{\mathbf{I}_a \times \dot{\mathbf{I}}_a} : \bar{L}^{n-16k}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2} & \longrightarrow & K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1) \\ \downarrow & & \downarrow \\ \zeta_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2} : L^{n-16k}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2} & \longrightarrow & K((\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z}, 1). \end{array}$$

Quaternionic structure theorem

Assume $n = 2^\ell - 2$, $n - 16k = 2^\sigma - 2$, $\ell \ge 11$, $\sigma \ge 5$. Assume that an element $x \in Imm^{sf}(n-k,k)$ admits a compression of order $q = 2^{\sigma-1} - 2$. Then the element $v = \delta^{[4]}_k \circ \delta^{[3]}_k \circ \delta^{[2]}_k(x) \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k)$ be represented by a triple (g, Ψ, η_N), which admits a quaternionic structure.

Solution of the Kervaire invariant problem (this is a correction of a mistake by the author, which is find-out by P.Landweber in arXiv:1001.4760)

Consider the codimension 8 submaniflod

$$N^{n-8k-8}_{\circ} \subset N^{n-8k},$$

which represents the Euler class $\eta^*_{a\times\dot{a}}(\tau'_{a\times\dot{a}})$,

$$\tau'_{a\times\dot{a}} \in H^8(K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)).$$

Define the standard mapping

$$\omega_{\mathbb{Z}/4} : K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1) \to K(\mathbb{Z}/4, 1).$$

Define a codimension 2 submanifold

$$N^{n-8k-10}_{\circ\circ} \subset N^{n-8k-8}_{\circ},$$

which represents the Euler class

$$[(\eta_{a\times\dot{a}} \circ \omega_{\mathbb{Z}/4})^*(\tau_{\mathbb{Z}/4})]^{op},$$

$$\tau_{\mathbb{Z}/4} \in H^2(K(\mathbb{Z}/4, 1)).$$

Consider the codimension 16 submaniflod

$$L_{\circ}^{n-16k-16} \subset L^{n-16k},$$

which represents the self-intersection manifold of the immersion $g|_{N_{\circ}^{n-8k-8}}$.

This submanifold admits the decomposition

$$L_{\mathbf{Q}\times\mathbb{Z}/4\circ}^{n-16k-16} \cup L_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2\circ}^{n-16k-16},$$

Consider the codimension 20 submaniflod

$$L_{\circ\circ}^{n-16k-20} \subset L^{n-16k},$$

which represents the self-intersection maniflod of $g|_{N_{\circ\circ}^{n-8k-10}}$. This submanifold admits the decomposition

$$L_{\mathbf{Q}\times\mathbb{Z}/4\circ\circ}^{n-16k-20} \cup L_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2\circ\circ}^{n-16k-20}.$$

Consider the element

$$\zeta_{a\times\dot{a},*}^{loc}([\bar{L}_{\mathbf{Q}\times\mathbb{Z}/4\circ}^{n-16k-16}]) \in D_{n-16k-16}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]),$$

for which the decomposition over the standard basis of the group

$$D_{n-16k-16}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) = H_{n-16k-16}(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1))$$

contains not more then the only non-trivial element, which is determined by the coefficient of the generator $t_{a,j} \otimes t_{\dot{a},j}$, $j = \frac{n-16k-16}{2}$.

Consider the element

$$\eta_{a\times\dot{a},*}^{loc}([\bar{L}_{\mathbf{Q}\times\mathbb{Z}/4\circ\circ}^{n-16k-20}]) \in D_{n-16k-20}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]), \qquad (5)$$

for which the decomposition over the standard basis of the group

$$D_{n-16k-20}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) = H_{n-16k-20}(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1))$$

contains the only two non-trivial elements, which are determined by the coefficient of the sum of the generators $t_{a,j} \otimes t_{\dot{a},j-4} + t_{a,j-4} \otimes t_{\dot{a},j}$.

Consider the element z with integer local coefficients

$$z = \eta^{loc,\mathbb{Z}}_{a\times\dot{a},*}([\bar{L}^{n-16k-20}_{\mathbf{Q}\times\mathbb{Z}/4\circ\circ}]) \in D_{n-16k-20}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}[\mathbb{Z}/2]), \qquad (6)$$

which is projected into (5) by means of the modulo 2 reduction of the coefficients. If the element (6) is of the order 2 then $\Theta^{sf}(x) = 0$.

Define the standard mapping

$$\omega_{\mathbf{Q}} : K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, 1) \to S^{4N+3}/\mathbf{Q} \subset K(\mathbf{Q}, 1), \quad N >> n.$$

This mapping induces the mapping

$$L^{n-16k-16}_{\mathbf{Q}\times\mathbb{Z}/4\circ} \to S^{4N+3}/\mathbf{Q}.$$

The submanifold $L^{n-16k-20}_{\mathbf{Q}\times\mathbb{Z}/4\circ\circ} \subset L^{n-16k-16}_{\mathbf{Q}\times\mathbb{Z}/4\circ}$ is the preimage of the standard subspace

$$S^{4N-1}/\mathbf{Q} \subset S^{4N+3}/\mathbf{Q}.$$

The fundamental class of the submanifold

$$\bar{L}^{n-16k-20}_{\mathbf{Q}\times\mathbb{Z}/4\circ\circ} \subset \bar{L}^{n-16k-16}_{\mathbf{Q}\times\mathbb{Z}/4\circ}$$

represents the homology Euler class

$$-[(\eta_{a\times\dot{a}} \circ \bar{\omega}_{\mathbf{Q}})^*(\tau_{\mathbb{Z}/4})]^{op}, \qquad \bar{\omega}_{\mathbf{Q}} = \omega_{\mathbb{Z}/4}.$$

The fundamental class of the submanifold

$$\bar{L}^{n-16k-20}_{\mathbf{Q}\times\mathbb{Z}/4\circ\circ} \subset N^{n-8k-8}_{\circ} \qquad (7)$$

represents the homology Euler class

$$[\eta^*_{a\times\dot{a}}((\tau'_{a\times\dot{a}})^k)(\eta_{a\times\dot{a}} \circ \omega_{\mathbb{Z}/4})^*(\tau^2_{\mathbb{Z}/4})]^{op}.$$

By the Herbert theorem (with integer local coefficients) the fundamental class of the submanifold (7) represents the homology Euler class

$$-[\eta^*_{a\times\dot{a}}((\tau'_{a\times\dot{a}})^k)(\eta_{a\times\dot{a}} \circ \omega_{\mathbb{Z}/4})^*(\tau^2_{\mathbb{Z}/4})]^{op}.$$

Therefore we get

$$0 = 2[\bar{L}^{n-16k-20}_{\mathbf{Q}\times\mathbb{Z}/4\circ\circ}] \in H_{n-16k-20}(N^{n-8k-8}_{\circ}; \mathbb{Z}[\mathbb{Z}/2]).$$

In particular, the element (6) is of the order 2 and $\Theta^{sf}(x) = 0$.

The Main Theorem is proved.