

Geometric approach to stable homotopy groups of spheres: The Kervaire invariant

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Let

$$f : M^{n-1} \looparrowright \mathbb{R}^n, \quad n = 4k + 2, \quad n \geq 2$$

be a smooth generic immersion of a closed manifold of codimension 1,

$$g : N^{n-2} \looparrowright \mathbb{R}^n$$

be the immersion of the double points intersection of g .

The **Kervaire invariant** of f is defined by

$$\Theta^{sf}(f) = \langle \eta_N^{\frac{n-2}{2}}; [N^{n-2}] \rangle,$$

where $\eta_N = w_2(N^{n-2})$ is the second normal Stiefel-Whitney class of N^{n-2} .

In particular, if $n = 2$, $\Theta_{sf}(f)$ is the parity of the number of self-intersection points of the regular curve f on the plane \mathbb{R}^2 .

Let us denote by $Imm^{sf}(n-1, 1)$ the cobordism group of immersions in the codimension 1 of (non-oriented) closed $(n-1)$ -manifold (" sf " stands for skew-framed).

Theorem

The Kervaire invariant is a well-defined homomorphism:

$$\Theta_{sf} : Imm^{sf}(4k + 1, 1) \longrightarrow \mathbb{Z}/2.$$

1. This homomorphism is trivial if $4k + 2 \neq 2^l - 2$, $l \geq 2$.
2. For $k = 0, 1, 3, 7, 15$ the homomorphism Θ^{sf} is an epimorphism.

Part 1 is the Theorem by W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math., (2) 90 (1969) 157-186. A special case of this theorem was discovered by M. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv., 34 (1960) 257-270.

Part 2 (in the case $k = 7$) was proved by M.E. Mahowald and M.C. Tangora, *Some differentials in the Adams spectral sequence*, Topology 6 (1967), 349-369. The case $k = 15$ was proved by M. G. Barratt, J. D. S. Jones and M. E. Mahowald, *Relations amongst Toda brackets and the Kervaire invariant in dimension 62*, J.London Math.Soc. (2) 30 (1984), no. 3, 533-550.

Main Theorem

There exists an integer l_0 , such that for an arbitrary $l \geq l_0$, the Kervaire invariant

$$\Theta^{sf} : Imm^{sf}(2^l - 3, 1) \longrightarrow \mathbb{Z}/2$$

is the trivial homomorphism.

The proof of this theorem is based on the approach proposed by P.J.Eccles *Codimension One Immersions and the Kervaire Invariant One Problem*, Math. Proc. Cambridge Phil. Soc., vol.90 (1981) 483-493.

Remark

The Main Theorem is a particular case of the Hill-Hopkins-Ravenel theorem ($l_0 = 8$).

A commutative diagram that we use to define dihedral structure of self-intersection manifold of skew-framed immersions:

$$\begin{array}{ccc}
 \text{Imm}^{sf}(n-1, 1) & \xrightarrow{J_k^{sf}} & \text{Imm}^{sf}(n-k, k) \\
 \downarrow \delta_k^{[2]} & & \downarrow \delta_k^{[2]} \\
 \text{Imm}^{\mathbb{Z}/2^{[2]}}(n-2, 2) & \xrightarrow{J_k^{[2]}} & \text{Imm}^{\mathbb{Z}/2^{[2]}}(n-2k, 2k).
 \end{array}$$

A commutative diagram that we use to define the Kervaire invariant in the codimension k :

$$\begin{array}{ccc}
 \text{Imm}^{sf}(n-k, k) & \xrightarrow{\Theta_k^{sf}} & \mathbb{Z}/2 & \frac{\text{skew-framed}}{\text{immersions}} \\
 \downarrow \delta_k^{[2]} & & \parallel & \\
 \text{Imm}^{\mathbb{Z}/2^{[2]}}(n-2k, 2k) & \xrightarrow{\Theta_k^{[2]}} & \mathbb{Z}/2 & \frac{\text{dihedral-framed}}{\text{immersions}}.
 \end{array}$$

Structural groups of immersions

Let us consider the following collection of $(d - 1)$ sets

$$\Upsilon_d, \Upsilon_{d-1}, \dots, \Upsilon_2,$$

where each set consists of proper coordinate subspaces of $\mathbb{R}^{2^{d-1}}$.

The set of the subspaces

$$\Upsilon_i, \quad 2 \leq i \leq d,$$

(we will use only the case $d \leq 5$) consists of 2^{i-1} coordinate subspaces generated by the basis vectors:

$$((\mathbf{e}_1, \dots, \mathbf{e}_{2^{d-i}}), \dots, (\mathbf{e}_{2^{d-1}-2^{d-i}+1}, \dots, \mathbf{e}_{2^{d-1}})).$$

Let us denote by $\mathbb{Z}/2^{[d]}$ the subgroup of the group

$$\mathbb{Z}/2 \wr \Sigma_{2^{d-1}} \subset O(2^{d-1})$$

under the following condition:

– the transformation

$$\mathbb{Z}/2^{[d]} \times \mathbb{R}^{2^{d-1}} \rightarrow \mathbb{R}^{2^{d-1}}$$

admits the invariant collection of sets

$$\Upsilon_d, \Upsilon_{d-1}, \dots, \Upsilon_2.$$

In particular, in the case $d = 2$ we get that Υ_2 contains only one collection of subspaces and this collection is $((\mathbf{e}_1), (\mathbf{e}_2))$. Therefore $\mathbb{Z}/2^{[2]}$ is the dihedral group.

$$\begin{array}{ccc}
Imm^{\mathbb{Z}/2^{[2]}}(n-2, 2) & \longrightarrow & Imm^{\mathbb{Z}/2^{[2]}}(n-2k, 2k) \\
\downarrow \delta^{[3]} & & \downarrow \delta_k^{[3]} \\
Imm^{\mathbb{Z}/2^{[3]}}(n-4, 4) & \xrightarrow{J_k^{[3]}} & Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k) \\
\downarrow \delta^{[4]} & & \downarrow \delta_k^{[4]} \\
Imm^{\mathbb{Z}/2^{[4]}}(n-8, 8) & \xrightarrow{J_k^{[4]}} & Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k) \\
\downarrow \delta^{[5]} & & \downarrow \delta_k^{[5]} \\
Imm^{\mathbb{Z}/2^{[5]}}(n-16, 16) & \xrightarrow{J_k^{[5]}} & Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)
\end{array}$$

$$\begin{array}{ccc}
Imm^{\mathbb{Z}/2^{[2]}}(n-2k, 2k) & \longrightarrow & \mathbb{Z}/2 \\
\downarrow \delta_k^{[3]} & & \parallel \\
Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k) & \xrightarrow{\Theta_k^{[3]}} & \mathbb{Z}/2 \\
\downarrow \delta_k^{[4]} & & \parallel \\
Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k) & \xrightarrow{\Theta_k^{[4]}} & \mathbb{Z}/2 \\
\downarrow \delta_k^{[5]} & & \parallel \\
Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k) & \xrightarrow{\Theta_k^{[5]}} & \mathbb{Z}/2
\end{array}$$

$$\begin{array}{ccc}
\mathbf{I}_b \times \dot{\mathbf{I}}_b & \subset & \mathbb{Z}/2^{[2]} & \frac{\text{abelian}}{\text{structure}} \\
\downarrow & & i^{[3]} \downarrow & \\
\mathbf{I}_a \times \dot{\mathbf{I}}_b & \subset & \mathbb{Z}/2^{[3]} & \frac{\text{bicyclic}}{\text{structure}} \\
\downarrow & & i^{[4]} \downarrow & \\
\mathbf{I}_a \times \dot{\mathbf{I}}_a & \subset & \mathbb{Z}/2^{[4]} & \frac{\text{bicyclic}}{\text{structure}} \\
\downarrow & & i^{[5]} \downarrow & \\
\mathbf{Q} \times \mathbb{Z}/4 & \subset & \mathbb{Z}/2^{[5]} & \frac{\text{quaternionic}}{\text{structure}}
\end{array}$$

Dihedral group

The dihedral group (of the order 8) $\mathbb{Z}/2^{[2]} \subset O(2)$:

$$\{a, b \mid a^4 = b^2 = e, [a, b] = a^2\}.$$

Let $\{\mathbf{f}_1, \mathbf{f}_2\}$ be the standard base of the plane \mathbb{R}^2 . The element a is represented by the rotation through the angle $\frac{\pi}{2}$:

$$f_1 \mapsto f_2; \quad f_2 \mapsto -f_1.$$

The element b is represented by the permutation of the base vectors

$$f_1 \mapsto f_2; \quad f_2 \mapsto f_1.$$

Elementary 2-group

The elementary subgroup $\mathbf{I}_b \times \dot{\mathbf{I}}_b \subset \mathbb{Z}/2^{[2]}$ of the rank 2:

$$\{a^2, b \mid a^4 = b^2 = e, [a^2, b] = e\}.$$

This group preserves the vectors $\mathbf{f}_1 + \mathbf{f}_2, \mathbf{f}_1 - \mathbf{f}_2$.

The group $(\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}$

Define the automorphism

$$\chi^{[2]} : \mathbf{I}_b \times \dot{\mathbf{I}}_b \rightarrow \mathbf{I}_b \times \dot{\mathbf{I}}_b$$

by the external conjugation of the subgroup $\mathbf{I}_b \times \dot{\mathbf{I}}_b \subset \mathbb{Z}/2^{[2]}$ by the element $ab \in \mathbb{Z}/2^{[2]}$.

Define the group

$$(\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \tag{1}$$

as the factorgroup of the group $(\mathbf{I}_b \times \dot{\mathbf{I}}_b) * \mathbb{Z}$ by the relation $zxz^{-1} = \chi^{[2]}(x)$, where $z \in \mathbb{Z}$ is the standard generator, $x \in \mathbf{I}_b \times \dot{\mathbf{I}}_b$ is an arbitrary element. The group (1) is a particular example of a semi-direct product of groups $A \rtimes_{\phi} B$, $A = \mathbf{I}_b \times \dot{\mathbf{I}}_b$, $B = \mathbb{Z}$, where $\phi : B \rightarrow \text{Aut}(A)$.

The classifying Eilenberg-Mac Lane space $K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1)$

Take a skew-product of the standard circle S^1 and the space $K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1)$, the shift $K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \rightarrow K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1)$, of the cyclic covering is given by the automorphism $\chi^{[2]}$. The standard inclusion

$$K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \subset K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1).$$

is well defined.

Homology with local coefficients

Define the group

$$H_i(K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]).$$

Consider the group ring $\mathbb{Z}/2[\mathbb{Z}/2] = \{a + bt\}$, $a, b \in \mathbb{Z}/2$, $t \in \mathbb{Z}/2$. Consider the following local system of the coefficients, associated with the homomorphism $\rho : (\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/2[\mathbb{Z}/2])$. Assume $x \in (\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}$. Define the automorphism $\rho(x)$ of $\mathbb{Z}/2[\mathbb{Z}/2]$ by the multiplication on the generator t , if $p_{b \times \dot{b}}(x) \equiv 1 \pmod{2}$ and by the identity automorphism, if $p_{b \times \dot{b}}(x) \equiv 0 \pmod{2}$, where

$$p_{b \times \dot{b}} : (\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \rightarrow \mathbb{Z}.$$

Splitting of homology

Define the subgroup

$$H_i(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1); \mathbb{Z}/2) \cong D_i(\mathbf{I}_b \times \dot{\mathbf{I}}_b; \mathbb{Z}/2[\mathbb{Z}/2]) \subset \\ H_i(K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])$$

by the following formula (is induced by the standard inclusion):

$$\text{Im}(H_i(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1); \mathbb{Z}/2[\mathbb{Z}/2]) \rightarrow H_i(K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])).$$

Define the splitting

$$\Delta^{[2]} : H_i(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1); \mathbb{Z}/2[\mathbb{Z}/2]) \rightarrow D_i(\mathbf{I}_b \times \dot{\mathbf{I}}_b; \mathbb{Z}/2[\mathbb{Z}/2])$$

by the formula $\Delta^{[2]}(X + Yt) = X + \chi_*^{[2]}(Y)$.

Reduction of the dihedral framing

Define the epimorphism

$$\Phi^{[2]} : (\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \rightarrow \mathbb{Z}/2^{[2]}$$

by the formula: $\Phi^{[2]}(z) = ab$ (the element ab corresponds to the reflection of the first vector of the standard base), the restriction

$$\Phi^{[2]}|_{\mathbf{I}_b \times \dot{\mathbf{I}}_b \times \{0\}} : \mathbf{I}_b \times \dot{\mathbf{I}}_b \subset \mathbb{Z}/2^{[2]}$$

is the standard inclusion.

Reduction of the characteristic class

Define

$$(\Phi^{[2]})^*(\tau_{[2]}) = \tau_{b \times \dot{b}},$$

where $\tau_{b \times \dot{b}} \in H^2(K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1))$, $\tau_{[2]} \in H^2(K(\mathbb{Z}^{[2]}, 1))$.

Definition of abelian structure

$$x = [(f, \Xi, \kappa_M)] \in Imm^{sf}(n-k, k),$$

- $f : M^{n-k} \looparrowright \mathbb{R}^n$,
- κ_M is a line bundle over M^{n-k} ,
- Ξ is a skew-framing of the normal bundle of f , i.e. an isomorphism $\Xi : \nu_f = k\kappa_M$.

$$y = \delta_k^{[2]}(x) = [(g, \Psi, \eta_N)] \in Imm^{\mathbb{Z}^{[2]}}(n-2k, 2k),$$

- $g : N^{n-2k} \looparrowright \mathbb{R}^n$,
- η_N is a $\mathbb{Z}/2^{[2]}$ -bundle over N^{n-2k} ,
- Ψ is a dihedral framing of the normal bundle of g , i.e. an isomorphism $\Psi : \nu_g = k\eta_N$.

We say that (f, Ξ, κ_M) admits an abelian structure if the cobordism class $y = [(g, \Psi, \eta_N)] = \delta_k^{[2]}(x)$ satisfies the following conditions:

1. There exists a mapping

$$\eta_{b \times \dot{b}} : N^{n-2k} \rightarrow K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1).$$

2. The restrictions of the mapping $\eta_{b \times \dot{b}, N}$ on the submanifold

$$N_{\eta^{7k}}^{n-16k} \subset N^{n-2k}, \tag{2}$$

which represents the (homology) Euler class $[\eta_N^{7k}]^{op} \in H_{n-16k}(N^{n-2k})$ of the bundle $7k\eta_N$, is given by the composition:

$$\eta_{b \times \dot{b}} : N_{\eta^{7k}}^{n-16k} \rightarrow K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \rightarrow K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1).$$

3. The restriction of the characteristic mapping η_N on the submanifold (2) is given by the composition $\Phi^{[2]} \circ \eta_{b \times \dot{b}, N}$:

$$N_{\eta^{7k}}^{n-16k} \rightarrow K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \rightarrow K((\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z}, 1) \rightarrow K(\mathbb{Z}/2^{[2]}, 1).$$

Example 1

Let a skew-framed immersion (f, Ξ, κ_M) , $f : M^{n-k} \looparrowright \mathbb{R}^n$ be represented an element $x \in Imm^{sf}(n-k, k)$, $n > 16k$ and be a $\mathbf{I}_b \times \dot{\mathbf{I}}_b$ -framed immersion. Then the skew-framed immersion (f, Ξ, κ_M) admits an abelian structure.

Example 2

Let a skew-framed immersion (f, Ξ, κ_M) , $f : M^{n-k} \looparrowright \mathbb{R}^n$ be represented an element $x \in Imm^{sf}(n-k, k)$, $n > 16k$ and be a $(\mathbf{I}_b \times \dot{\mathbf{I}}_b)_{\chi^{[2]}} \int \mathbb{Z}$ -framed immersion, and the restriction of this immersion on the submanifold

$$N_{\eta^{7k}}^{n-16k} \subset N^{n-2k}$$

is a $\mathbf{I}_b \times \dot{\mathbf{I}}_b$ -framed immersion. Then the skew-framed immersion (f, Ξ, κ_M) admits an abelian structure.

Define the codimension 2 submanifold

$$N_{\circ}^{n-2k-2} \subset N^{n-2k},$$

which represents the Euler class $\eta_{b \times \dot{b}}^*(\tau_{b \times \dot{b}})$. Define the codimension 16 submanifold

$$N_{\eta^{7k_{\circ}}}^{n-16k-16} \subset N_{\eta^{7k}}^{n-16k},$$

which represents the Euler class $\eta_{b \times \dot{b}}^*(\tau_{b \times \dot{b}}^4)$. Define the element

$$\eta_{b \times \dot{b}, * }^{loc}([N_{\eta^{7k_{\circ}}}^{n-16k-16}]) \in D_{n-16k-16}(\mathbf{I}_b \times \dot{\mathbf{I}}_b; \mathbb{Z}/2[\mathbb{Z}/2]). \quad (3)$$

Lemma-Definition

Let a skew-framed immersion (f, Ξ, κ_M) be admitted an abelian structure. Then for the element $y = \delta_k^{[2]}(x) = [(g, \Psi, \eta_N)] \in Imm^{\mathbb{Z}^{[2]}}(n-2k, 2k)$, The decomposition of the image of the homology class (3) over the standard basis of the group

$$D_{n-16k-16}(\mathbf{I}_b \times \dot{\mathbf{I}}_b; \mathbb{Z}/2[\mathbb{Z}/2]) = H_{n-16k-16}(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1))$$

contains not more then the only non-trivial element, which is determined by the coefficient of the generator $t_{b,j} \otimes t_{\dot{b},j}$, $j = \frac{n-16k-16}{2}$. This coefficient coincides with the Kervaire invariant for the $\mathbb{Z}/2^{[2]}$ -framed immersion (g, Ψ, η_N) .

Compression

A skew-framed cobordism class

$$x = [(f, \Xi, \kappa_M)] \in Imm^{sf}(n-k, k)$$

admits a compression of order q , if this class is represented by a triple, such that

$$\kappa_M = I \circ \kappa_{(q)},$$

$$\kappa_{(q)} : M^{n-k} \rightarrow \mathbb{RP}^{n-k-q-1}, I : \mathbb{RP}^{n-k-q-1} \subset \mathbb{RP}^\infty.$$

Compression theorem

For an arbitrary q there exists an integer $l_0 = l_0(q)$, such that an arbitrary element $x \in Imm^{sf}(2^l - 3, 1)$, $l \geq l_0$, admits a compression of order q .

Abelian structure theorem

Let $2^l - 16k = n - 16k = 2^\sigma - 2$, $k \equiv 0 \pmod{2}$, ($l \gg \sigma \geq 5$). Assume that the element $x \in Imm^{sf}(n-k, k)$ admits a compression of order $q = 2^{\sigma-1} - 2$. Then the element x be represented by a triple (f, Ξ, κ_M) , which admits an abelian structure.

Cyclic group

The cyclic index 2 subgroup of order 4:

$$\mathbf{I}_a = \{a \mid a^4 = e\} \subset \mathbb{Z}/2^{[2]}.$$

Bicyclic subgroups

The bicyclic index 2^{11} subgroup of the order 16:

$$\mathbf{I}_a \times \dot{\mathbf{I}}_a \subset \mathbb{Z}/2^{[4]}.$$

Denote the generators by

$$t_{a,j} \in H_j(K(\mathbf{I}_a, 1)); \quad t_{\dot{a},j} \in H_j(K(\dot{\mathbf{I}}_a, 1)), \quad j \equiv 1 \pmod{2}.$$

Define automorphism of order 2:

$$\chi^{[4]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \rightarrow \mathbf{I}_a \times \dot{\mathbf{I}}_a.$$

Define the group

$$(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}.$$

Define the natural homomorphism

$$\Phi^{[4]} : (\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z} \rightarrow \mathbb{Z}/2^{[4]}.$$

The classifying space $K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)$ is the semi-direct product of the standard circle S^1 and the space $K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1)$.

Homology with local coefficients

Define the group

$$H_i(K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]).$$

Define the subgroup

$$\begin{aligned} H_i(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2) &\cong D_i(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) \subset \\ &H_i(K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]). \end{aligned}$$

Define the splitting

$$\Delta^{[4]} : H_i(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1); \mathbb{Z}/2[\mathbb{Z}/2]) \rightarrow D_i(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]).$$

Structural groups of framings

$$(\mathbf{I}_b \times \dot{\mathbf{I}}_b) \int_{\chi^{[2]}} \mathbb{Z} \xrightarrow{\Phi^{[2]}} \mathbb{Z}/2^{[2]}$$

$$i_{b \times b, \mathbf{I}_a \times \dot{\mathbf{I}}_b} \downarrow \quad i^{[3]} \downarrow$$

$$(\mathbf{I}_a \times \dot{\mathbf{I}}_b) \int_{\chi^{[3]}} \mathbb{Z} \xrightarrow{\Phi^{[3]}} \mathbb{Z}/2^{[3]}$$

$$i_{\mathbf{I}_a \times \dot{\mathbf{I}}_b, \mathbf{I}_a \times \dot{\mathbf{I}}_a} \downarrow \quad i^{[4]} \downarrow$$

$$(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z} \xrightarrow{\Phi^{[4]}} \mathbb{Z}/2^{[4]}.$$

$$x \in \text{Imm}^{sf}(n-k, k),$$

$$y = \delta^{[2]}(x) \in \text{Imm}^{\mathbb{Z}/2^{[2]}}(n-2k, 2k),$$

$$z = \delta^{[3]} \circ \delta^{[2]} \in \text{Imm}^{\mathbb{Z}/2^{[3]}}(n-4k, 4k),$$

$$u = \delta^{[4]} \circ \delta^{[3]} \circ \delta^{[2]}(x),$$

$$u = [(h, \Lambda, \zeta_L)] \in \text{Imm}^{\mathbb{Z}/2^{[4]}}(n-8k, 8k).$$

$$-h : L^{n-8k} \looparrowright \mathbb{R}^n,$$

$$-\zeta_L \text{ is a } \mathbb{Z}/2^{[4]}\text{-bundle over } L^{n-8k},$$

$-\Lambda$ is a an 8-dimensional $\mathbb{Z}/4^{[4]}$ -framing of the normal bundle of h , i.e. an isomorphism $\Lambda : \nu_h \simeq k\zeta_L$.

Define the codimension 8 submaniflod

$$L_{\circ}^{n-8k-8} \subset L^{n-8k},$$

which represents the Euler class $\zeta_{a \times \dot{a}}^*(\tau'_{a \times \dot{a}})$,

$$\tau'_{a \times \dot{a}} \in H^8(K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)),$$

$$\tau'_{a \times \dot{a}} = \Phi^{[4]*}(\tau_{[4]}), \quad \tau_{[4]} \in H^8(K(\mathbb{Z}/2^{[4]}, 1))$$

Define the codimension 16 submaniflod

$$L_{\zeta^k \circ}^{n-16k-16} \subset L_{\zeta^k}^{n-16k},$$

which represents the Euler class $\zeta_{a \times \dot{a}}^*(\tau'_{a \times \dot{a}})$. Define the element

$$\zeta_{a \times \dot{a}, *}^{loc}([L_{\zeta^k \circ}^{n-16k-16}]) \in D_{n-16k-16}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]). \quad (4)$$

Definition of bicyclic structure

We say that a triple (g, Ψ, η_N) , which represents an element in $Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$, admits a bicyclic structure if for $\delta^{[4]}(z) = [(h, \Lambda, \zeta_L)]$ there exists a mapping

$$\zeta_{a \times \dot{a}} : L^{n-8k} \rightarrow K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1),$$

for which the decomposition of the image of the homology class $\zeta_{a \times \dot{a}, *}^{loc}([L_{\zeta^{k_o}}^{n-16k-16}])$ over the standard basis of the group

$$D_{n-16k-16}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) = H_{n-16k-16}(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1))$$

contains not more than the only non-trivial element, which is determined by the coefficient of the generator $t_{a,j} \otimes t_{\dot{a},j}$, $j = \frac{n-16k-16}{2}$. This coefficient coincides with the Kervaire invariant for the $\mathbb{Z}/2^{[4]}$ -framed immersion (h, Λ, ζ_L) .

Bicyclic structure theorem

Assume that the dimensional restriction of Abelian structure theorem is satisfied and the element $x \in Imm^{sf}(n-k, k)$ admits a compression of order $q = 2^{\sigma-1} - 2$. Then the cobordism class $z = \delta^{[3]} \circ \delta^{[2]} \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$ be represented by a triple (g, Ψ, η_N) , which admits a bicyclic structure.

Quaternionic groups

The quaternionic group of the order 8:

$$\mathbf{Q} = \{\mathbf{i}, \mathbf{j}, \mathbf{k} \mid \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\}.$$

Define the subgroups

$$i_{\mathbf{Q} \times \mathbb{Z}/4} : \mathbf{Q} \times \mathbb{Z}/4 \subset \mathbb{Z}/2^{[5]},$$

$$i_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2 \subset \mathbb{Z}/2^{[5]}.$$

Define the involutions

$$\chi^{[5]} : \mathbf{Q} \times \mathbb{Z}/4 \rightarrow \mathbf{Q} \times \mathbb{Z}/4,$$

$$\chi^{[5]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2 \rightarrow \mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2.$$

Define groups

$$(\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z},$$

$$(\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z}$$

Define the following homomorphisms:

$$\Phi^{[5]} : (\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z} \rightarrow \mathbb{Z}/2^{[5]},$$

$$\Phi^{[5]} : (\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z} \rightarrow \mathbb{Z}/2^{[5]}.$$

Commutative diagrams are well defined:

$$\begin{array}{ccc} (\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z} & \xrightarrow{\Phi^{[4]} \times \Phi^{[4]}} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\ i_{\mathbf{Q} \times \mathbb{Z}/4} \downarrow & & i_{[5]} \downarrow \\ (\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z} & \xrightarrow{\Phi^{[5]}} & \mathbb{Z}/2^{[5]}. \end{array}$$

$$\begin{array}{ccc} (\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z} & \xrightarrow{\Phi^{[4]} \times \Phi^{[4]}} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\ i_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2} \downarrow & & i_{[5]} \downarrow \\ (\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z} & \xrightarrow{\Phi^{[5]}} & \mathbb{Z}/2^{[5]}. \end{array}$$

Definition of quaternionic structure

Let a $\mathbb{Z}/2^{[4]}$ -framed immersion (g, Ψ, η_N) , $g : N^{n-8k} \looparrowright \mathbb{R}^n$ be represented an element $v \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k)$. We say that v admits a quaternionic structure if there exists a mapping

$$\eta_{\mathbf{I}_a \times \dot{\mathbf{I}}_a, N} : N^{n-8k} \rightarrow K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1),$$

which satisfies conditions of a bicyclic structure and the following conditions:

Let a $\mathbb{Z}/2^{[5]}$ -framed immersion (h, Λ, ζ_L) , $h : L^{n-16k} \looparrowright \mathbb{R}^n$, be the immersion of self-intersection points of g and be represented an element $w = \delta^{[5]}(v) \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)$. Let the manifold L^{n-16k} be represented by the following disjoint union of two components:

$$L^{n-16k} = L_{\mathbf{Q} \times \mathbb{Z}/4}^{n-16k} \cup L_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2}^{n-16k},$$

and the following diagrams of canonical 2-sheeted coverings are commutative:

$$\begin{array}{ccc}
\eta_{\mathbf{I}_a \times \dot{\mathbf{I}}_a} : \bar{L}_{\mathbf{Q} \times \mathbb{Z}/4}^{n-16k} & \longrightarrow & K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1) \\
\downarrow & & \downarrow \\
\zeta_{\mathbf{Q} \times \mathbb{Z}/4} : L_{\mathbf{Q} \times \mathbb{Z}/4}^{n-16k} & \longrightarrow & K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, 1), \\
\eta_{\mathbf{I}_a \times \dot{\mathbf{I}}_a} : \bar{L}_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2}^{n-16k} & \longrightarrow & K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1) \\
\downarrow & & \downarrow \\
\zeta_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2} : L_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2}^{n-16k} & \longrightarrow & K((\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z}, 1).
\end{array}$$

Quaternionic structure theorem

Assume $n = 2^\ell - 2$, $n - 16k = 2^\sigma - 2$, $\ell \geq 11$, $\sigma \geq 5$. Assume that an element $x \in Imm^{sf}(n - k, k)$ admits a compression of order $q = 2^{\sigma-1} - 2$. Then the element $v = \delta_k^{[4]} \circ \delta_k^{[3]} \circ \delta_k^{[2]}(x) \in Imm^{\mathbb{Z}/2^{[4]}}(n - 8k, 8k)$ be represented by a triple (g, Ψ, η_N) , which admits a quaternionic structure.

Solution of the Kervaire invariant problem (this is a correction of a mistake by the author, which is find-out by P.Landweber in arXiv:1001.4760)

Consider the codimension 8 submaniflod

$$N_{\circ}^{n-8k-8} \subset N^{n-8k},$$

which represents the Euler class $\eta_{a \times \dot{a}}^*(\tau'_{a \times \dot{a}})$,

$$\tau'_{a \times \dot{a}} \in H^8(K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)).$$

Define the standard mapping

$$\omega_{\mathbb{Z}/4} : K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1) \rightarrow K(\mathbb{Z}/4, 1).$$

Define a codimension 2 submanifold

$$N_{\circ\circ}^{n-8k-10} \subset N_{\circ}^{n-8k-8},$$

which represents the Euler class

$$\begin{aligned}
& [(\eta_{a \times \dot{a}} \circ \omega_{\mathbb{Z}/4})^*(\tau_{\mathbb{Z}/4})]^{op}, \\
& \tau_{\mathbb{Z}/4} \in H^2(K(\mathbb{Z}/4, 1)).
\end{aligned}$$

Consider the codimension 16 submanifold

$$L_{\circ}^{n-16k-16} \subset L^{n-16k},$$

which represents the self-intersection manifold of the immersion $g|_{N_{\circ}^{n-8k-8}}$.

This submanifold admits the decomposition

$$L_{\mathbf{Q} \times \mathbb{Z}/4\circ}^{n-16k-16} \cup L_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2\circ}^{n-16k-16},$$

Consider the codimension 20 submanifold

$$L_{\circ\circ}^{n-16k-20} \subset L^{n-16k},$$

which represents the self-intersection manifold of $g|_{N_{\circ\circ}^{n-8k-10}}$. This submanifold admits the decomposition

$$L_{\mathbf{Q} \times \mathbb{Z}/4\circ\circ}^{n-16k-20} \cup L_{\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2\circ\circ}^{n-16k-20}.$$

Consider the element

$$\zeta_{a \times \dot{a}, *}^{loc}([\bar{L}_{\mathbf{Q} \times \mathbb{Z}/4\circ}^{n-16k-16}]) \in D_{n-16k-16}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]),$$

for which the decomposition over the standard basis of the group

$$D_{n-16k-16}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) = H_{n-16k-16}(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1))$$

contains not more than the only non-trivial element, which is determined by the coefficient of the generator $t_{a,j} \otimes t_{\dot{a},j}$, $j = \frac{n-16k-16}{2}$.

Consider the element

$$\eta_{a \times \dot{a}, *}^{loc}([\bar{L}_{\mathbf{Q} \times \mathbb{Z}/4\circ\circ}^{n-16k-20}]) \in D_{n-16k-20}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]), \quad (5)$$

for which the decomposition over the standard basis of the group

$$D_{n-16k-20}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) = H_{n-16k-20}(K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1))$$

contains the only two non-trivial elements, which are determined by the coefficient of the sum of the generators $t_{a,j} \otimes t_{\dot{a},j-4} + t_{a,j-4} \otimes t_{\dot{a},j}$.

Consider the element z with integer local coefficients

$$z = \eta_{a \times \dot{a}, *}^{loc, \mathbb{Z}}([\bar{L}_{\mathbf{Q} \times \mathbb{Z}/4\circ\circ}^{n-16k-20}]) \in D_{n-16k-20}(\mathbf{I}_a \times \dot{\mathbf{I}}_a; \mathbb{Z}[\mathbb{Z}/2]), \quad (6)$$

which is projected into (5) by means of the modulo 2 reduction of the coefficients. If the element (6) is of the order 2 then $\Theta^{sf}(x) = 0$.

Define the standard mapping

$$\omega_{\mathbf{Q}} : K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, 1) \rightarrow S^{4N+3}/\mathbf{Q} \subset K(\mathbf{Q}, 1), \quad N \gg n.$$

This mapping induces the mapping

$$L_{\mathbf{Q} \times \mathbb{Z}/4\circ}^{n-16k-16} \rightarrow S^{4N+3}/\mathbf{Q}.$$

The submanifold $L_{\mathbf{Q} \times \mathbb{Z}/4\circ\circ}^{n-16k-20} \subset L_{\mathbf{Q} \times \mathbb{Z}/4\circ}^{n-16k-16}$ is the preimage of the standard subspace

$$S^{4N-1}/\mathbf{Q} \subset S^{4N+3}/\mathbf{Q}.$$

The fundamental class of the submanifold

$$\bar{L}_{\mathbf{Q} \times \mathbb{Z}/4\circ\circ}^{n-16k-20} \subset \bar{L}_{\mathbf{Q} \times \mathbb{Z}/4\circ}^{n-16k-16}$$

represents the homology Euler class

$$-[(\eta_{a \times \dot{a}} \circ \bar{\omega}_{\mathbf{Q}})^*(\tau_{\mathbb{Z}/4})]^{op}, \quad \bar{\omega}_{\mathbf{Q}} = \omega_{\mathbb{Z}/4}.$$

The fundamental class of the submanifold

$$\bar{L}_{\mathbf{Q} \times \mathbb{Z}/4\circ\circ}^{n-16k-20} \subset N_{\circ}^{n-8k-8} \quad (7)$$

represents the homology Euler class

$$[\eta_{a \times \dot{a}}^*((\tau'_{a \times \dot{a}})^k)(\eta_{a \times \dot{a}} \circ \omega_{\mathbb{Z}/4})^*(\tau_{\mathbb{Z}/4}^2)]^{op}.$$

By the Herbert theorem (with integer local coefficients) the fundamental class of the submanifold (7) represents the homology Euler class

$$-[\eta_{a \times \dot{a}}^*((\tau'_{a \times \dot{a}})^k)(\eta_{a \times \dot{a}} \circ \omega_{\mathbb{Z}/4})^*(\tau_{\mathbb{Z}/4}^2)]^{op}.$$

Therefore we get

$$0 = 2[\bar{L}_{\mathbf{Q} \times \mathbb{Z}/4\circ\circ}^{n-16k-20}] \in H_{n-16k-20}(N_{\circ}^{n-8k-8}; \mathbb{Z}[\mathbb{Z}/2]).$$

In particular, the element (6) is of the order 2 and $\Theta^{sf}(x) = 0$.

The Main Theorem is proved.