Quadratic difference forms and Lyapunov stability of 2D systems

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Outline

Introduction and motivation

$2D$ systems

$2D$ stability

Quadratic difference forms (QdFs) for $2D$ systems

Main result
Main points

• Trajectory-based approach to dynamics
Main points

- Trajectory-based approach to dynamics
- Multivariate polynomial algebra for dynamics and functionals
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Main result
Discrete $2D$ systems

‘Discrete $2D$ system’ means:

\[ \mathcal{B} = \{ w : \mathbb{Z}^2 \rightarrow \mathbb{R}^w \mid R(\sigma_1, \sigma_2)w = 0 \} \]

where

\[
(\sigma_1 w)(x_1, x_2) := w(x_1 + 1, x_2)
\]

\[
(\sigma_2 w)(x_1, x_2) := w(x_1, x_2 + 1)
\]

$R$: two-variable Laurent polynomial matrix with $w$ columns
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Algebraic properties of \( R \leftrightarrow \) properties of \( \mathcal{B} \)
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1D autonomy and stability

Time has one direction only: past → future
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⇒ ‘autonomy’ as

‘Past uniquely determines future’
1D autonomy and stability

Time has one direction only: past $\rightarrow$ future

$\leadsto$ 'autonomy' as

'Past uniquely determines future'

$\leadsto$ stability as

'Future trajectories die out'
**1D autonomy and stability**

Time has one direction only: \( \text{past} \rightarrow \text{future} \)

\[ \Rightarrow \text{‘autonomy’ as} \]

‘Past uniquely determines future’

\[ \Rightarrow \text{stability as} \]

‘Future trajectories die out’

...what is ‘past’ and ‘future’ for \( 2D \) systems?
2D autonomy

$S \in \mathbb{Z}^2$ characteristic for $\mathcal{B}$ if

$$[w_1, w_2 \in \mathcal{B} \text{ and } w_{1|S} = w_{2|S}] \implies [w_1 = w_2]$$

Note $\mathbb{Z}^2$ is characteristic for every $\mathcal{B}$... trivial!
2D autonomy

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\( \mathcal{B} \) is autonomous if \( \exists \) proper cone characteristic for it.

**Proper cone**: convex, closed, ‘pointed’, generated by linearly independent \( v_1, v_2 \in \mathbb{R}^2 \)
**2D autonomy**

\( S \in \mathbb{Z}^2 \) **characteristic** for \( \mathcal{B} \) if

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Note \( \mathbb{Z}^2 \) is characteristic for every \( \mathcal{B} \) ... trivial!

\( \mathcal{B} \) is **autonomous** if \( \exists \) **proper cone** characteristic for it.

Given \( \mathcal{B} = \ker R(\sigma_1, \sigma_2) \), characterisable algebraically in terms of properties of \( R(\xi_1, \xi_2) \).
2D stability

If characteristic proper cone $\mathcal{K} \leftrightarrow \text{'past'}$...
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...then $-\mathcal{K}$ is ‘future’.
2D stability

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Does $w$ die out in $-\mathcal{K}$ given arbitrary ‘initial conditions’ in $\mathcal{K}$?
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Only for finite-dimensional 2D systems.
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Does $w$ die out in $-\mathcal{K}$
given arbitrary ‘initial conditions’ in $\mathcal{K}$?

Only for finite-dimensional $2D$ systems.

$\infty$-dimensional $\mathcal{V} \implies$ unbounded ‘initial conditions’
can produce unbounded future.
$2D$ stability: ‘square’ infinite-dimensional case

$\delta(-K)$: generating lines
2D stability: ‘square’ infinite-dimensional case

\( \delta(-\mathcal{K}) \): generating lines

\( \delta(-\mathcal{K})^n \): strip around \( \delta(-\mathcal{K}) \)
2D stability: ‘square’ infinite-dimensional case

\( \delta(-K) \): generating lines

\( \delta(-K)^n \): strip around \( \delta(-K) \)

\( \mathcal{B} = \ker R(\sigma_1, \sigma_2) \) with \( R \in \mathbb{R}^{w \times w}[\xi_1, \xi_2] \) nonsingular is \( \mathcal{K} \)-stable if \( \exists \ n \in \mathbb{N}, \ n > 0 \) s.t.

\( w \in \mathcal{B}, \ w \) bounded in \( \delta(-K)^n \) \( \Rightarrow \) \( \lim_{(i, j) \in \mathcal{K}} \| w(i, j) \| = 0 \) \( \iff |i| + |j| \to +\infty \)

Algebraic characterization available (Valcher ’00)
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Quadratic difference forms (QdFs) for 2D systems

Main result
Four-variable polynomial matrices and QdFs

Let $k := (k_1, k_2)$, $\ell := (\ell_1, \ell_2)$, and $\zeta^k \eta^\ell := \zeta_1^{k_1} \zeta_2^{k_2} \eta_1^{\ell_1} \eta_2^{\ell_2}$.

Write $\Phi \in \mathbb{R}^{w \times w}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ as

$$
\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) = \Phi(\zeta, \eta) = \sum_{k,\ell} \zeta^k \Phi_{k,\ell} \eta^\ell
$$
Four-variable polynomial matrices and QdFs

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$$\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) = \Phi(\zeta, \eta) = \sum_{k, \ell} \zeta^k \Phi_{k, \ell} \eta^\ell$$

Induces $Q_\Phi : (\mathbb{R}^w)^{\mathbb{Z} \times \mathbb{Z}} \rightarrow (\mathbb{R})^{\mathbb{Z} \times \mathbb{Z}}$ as

$$Q_\Phi(w) := \sum_{k, \ell} (\sigma^k w)^\top \Phi_{k, \ell} (\sigma^\ell w)$$

where $\sigma^k := \sigma_1^{k_1} \sigma_2^{k_2}$, $\sigma^\ell := \sigma_1^{\ell_1} \sigma_2^{\ell_2}$. 
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Write \( \Phi \in \mathbb{R}^{w \times w}[\zeta_1, \zeta_2, \eta_1, \eta_2] \) as

\[
\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) = \Phi(\zeta, \eta) = \sum_{k, \ell} \zeta^k \Phi_{k, \ell} \eta^\ell
\]

Induces \( Q_\Phi : (\mathbb{R}^w)^{\mathbb{Z} \times \mathbb{Z}} \rightarrow (\mathbb{R})^{\mathbb{Z} \times \mathbb{Z}} \) as

\[
Q_\Phi(w) := \sum_{k, \ell} (\sigma^k w)^T \Phi_{k, \ell}(\sigma^\ell w)
\]

where \( \sigma^k := \sigma_1^{k_1} \sigma_2^{k_2} \), \( \sigma^\ell := \sigma_1^{\ell_1} \sigma_2^{\ell_2} \).

Calculus: \( Q_\psi(\cdot) := \sigma_1 Q_\Phi(\cdot) \)

\[
\sim \psi(\zeta, \eta) = \zeta_1 \Phi(\zeta, \eta) \eta_1
\]
Divergence of a QdF

Vector QdF:

\[ \text{col}(Q_{\Phi_1}, Q_{\Phi_2}) : (\mathbb{R}^w)^{Z \times Z} \rightarrow (\mathbb{R})^{Z \times Z} \times (\mathbb{R})^{Z \times Z} \]
Divergence of a QdF

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\[
(\text{div \ col}(Q_{\Phi_1}, Q_{\Phi_2}))(w) := \\
(Q_{\Phi_1}(w) - \sigma_1(Q_{\Phi_1}(w))) \\
+ (Q_{\Phi_2}(w) - \sigma_2(Q_{\Phi_2}(w)))
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Divergence of a QdF

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Polynomially:

\[
\nabla \Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) := (1 - \zeta_1 \eta_1) \psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2)
+ (1 - \zeta_2 \eta_2) \psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)
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Setting the stage

$$\text{Per}_2 := \{ \mathbf{v} \in (\mathbb{R}^w)^{\mathbb{Z} \times \mathbb{Z}} \mid \mathbf{v}(x_1, \cdot) \in (\mathbb{R}^w)^{\mathbb{Z}} \text{ periodic \ } \forall \ x_1 \in \mathbb{Z} \}$$

$$\text{Per}_1 := \{ \mathbf{v} \in (\mathbb{R}^w)^{\mathbb{Z} \times \mathbb{Z}} \mid \mathbf{v}(\cdot, x_2) \in (\mathbb{R}^w)^{\mathbb{Z}} \text{ periodic \ } \forall \ x_1 \in \mathbb{Z} \}$$

Trajectories periodic on horizontal/vertical lines.
Setting the stage

\( \text{Per}_2 := \{ \mathbf{v} \in (\mathbb{R}^w)^{\mathbb{Z} \times \mathbb{Z}} \mid \mathbf{v}(x_1, \cdot) \in (\mathbb{R}^w)^{\mathbb{Z}} \text{ periodic } \forall x_1 \in \mathbb{Z} \} \)

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Trajectories periodic on horizontal/vertical lines.

\( \mathcal{B} \cap \text{Per}_2 = \{ \mathbf{w} \in \mathcal{B} \mid \mathbf{w} \text{ periodic along } x_1\text{-direction} \} \)

Using Fourier argument:

\( \mathcal{B} = \ker R(\sigma_1, \sigma_2) \implies \mathcal{B} \cap \text{Per}_2 = \bigcup_{\omega \in \mathbb{R}} \ker R(e^{j\omega}, \sigma_2) \)
Main result: trajectory version

Let $\mathcal{B} = \ker R(\sigma_1, \sigma_2)$, $R \in \mathbb{R}^{w \times w}[\xi_1, \xi_2]$ nonsingular. The following are equivalent:

1. $\mathcal{B}$ is first-orthant-stable;

2. $\exists Q_\Phi = \text{col}(Q_{\Phi_1}, Q_{\Phi_2})$ and $Q_\Delta$ such that
   - $\text{div } Q_\Phi(w) = -Q_\Delta(w)$ $\forall w \in \mathcal{B}$;
   - $Q_{\Phi_1}(w), Q_\Delta(w) > 0$ for all $w \in \mathcal{B} \cap \text{Per}_2$, and $Q_{\Phi_2}(w), Q_\Delta(w) > 0$ for all $w \in \mathcal{B} \cap \text{Per}_1$. 
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1. $\mathcal{B}$ is first-orthant-stable;

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   - $\text{div } Q_\Phi(w) = - Q_\Delta(w)$ $\forall w \in \mathcal{B}$;
   - $Q_{\phi_1}(w), Q_\Delta(w) > 0$ for all $w \in \mathcal{B} \cap \text{Per}_2$, and $Q_{\phi_2}(w), Q_\Delta(w) > 0$ for all $w \in \mathcal{B} \cap \text{Per}_1$.

$\text{col}(Q_{\phi_1}, Q_{\phi_2})$ is Lyapunov function for $\mathcal{B}$. 
Main result: algebraic version

Let $\mathcal{B} = \ker R(\sigma_1, \sigma_2)$, $R \in \mathbb{R}^{w \times w}[\xi_1, \xi_2]$ nonsingular. Then 1) and 2) are equivalent with:

3. \( \exists \Phi_1, \Phi_2, Y, \Delta \in \mathbb{R}^{w \times w}[\zeta_1, \zeta_2, \eta_1, \eta_2] \), such that

\begin{align*}
\nabla \ \text{col}(\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2), \Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)) &= -\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) + R(\zeta_1, \zeta_2)^\top Y(\zeta_1, \zeta_2, \eta_1, \eta_2) \\
& \quad + Y(\eta_1, \eta_2, \zeta_1, \zeta_2)^\top R(\eta_1, \eta_2);
\end{align*}

\begin{align*}
\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) &> 0, \quad \mathcal{B} \cap \text{Per}_2 \\
\Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) &> 0, \quad \mathcal{B} \cap \text{Per}_1 \\
\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) &> 0, \quad \mathcal{B} \cap \text{Per}_i, \quad i = 1, 2.
\end{align*}
Main result: algebraic version

Let \( \mathcal{B} = \ker R(\sigma_1, \sigma_2) \), \( R \in \mathbb{R}^{w \times w}[\xi_1, \xi_2] \) nonsingular.

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\nabla \ \text{col}(\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2), \Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)) = \\
-\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) + R(\zeta_1, \zeta_2)^\top Y(\zeta_1, \zeta_2, \eta_1, \eta_2) \\
+ Y(\eta_1, \eta_2, \zeta_1, \zeta_2)^\top R(\eta_1, \eta_2); \\
\]

- \( \Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2)^\mathcal{B} \cap \text{Per}_2 > 0 \),
- \( \Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)^\mathcal{B} \cap \text{Per}_1 > 0 \),
- \( \Delta(\zeta_1, \zeta_2, \eta_1, \eta_2)^\mathcal{B} \cap \text{Per}_i > 0, \ i = 1, 2. \)

2D polynomial Lyapunov equation
References


Remarks and open questions

- Polynomial framework for dynamics & functionals
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- Properties/operations on dynamics/functionals
  \[\leftrightarrow\] properties/operations on polynomial representations
- Special role of variable “time” possible
- \(2D\) Lyapunov equation: solvability? algorithms?
- Parametric Lyapunov functions for robust stability?
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THANK YOU!