**Minkowski's Conjecture (?)**

a. For any lattice $\Lambda \subset \mathbb{R}^d$ of covolume 1, and any $\vec{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$,

$$(\mathrm{M}) \qquad \inf_{\vec{\lambda} \in \Lambda} \prod_1^d |\lambda_i - z_i| \leq \frac{1}{2^d}.$$

b. Equality holds in (M) iff $\Lambda = \mathbb{Z}^d$ and $z \in \mathbb{Z}^d + (1/2, \ldots, 1/2)$.

Proved by Minkowski in 1899 for $d = 2$.

To date has been proved for $d \leq 8$.

We will only discuss part (a.)

## Geometric reformulation

Let $N : \mathbb{R}^d \to \mathbb{R}, \quad N(\vec{x}) = \prod_1^d |x_i|$,

$$\mathcal{S} = \left\{ \vec{x} : N(\vec{x}) \leq \frac{1}{2^d} \right\}.$$

Then MC is equivalent to:

for any $\Lambda$, $\Lambda + \mathcal{S} = \mathbb{R}^d$.

Equivalently, for any $\vec{z}$ and $\Lambda$,

$$(\vec{z} + \Lambda) \cap \mathcal{S} \neq \varnothing.$$

A *unimodular grid in* $\mathbb{R}^d$ is a set of the form $\vec{z} + \Lambda$, where $\vec{z} \in \mathbb{R}^d$ and $\Lambda$ is a lattice of covolume 1.

The conjecture states that *any unimodular grid intersects the star-body* $S$.

**Dynamical reformulation**

Let $G_0 = \mathrm{SL}_d(\mathbb{R}) \ltimes \mathbb{R}^d$. $G_0$ acts transitively on unimodular grids. Stabilizer of $\mathbb{Z}^d$ is $\Gamma_0 = \mathrm{SL}_d(\mathbb{Z}) \ltimes \mathbb{Z}^d$.

Hence: $\{\text{unimodular grids}\} = G_0/\Gamma_0$.

This is a finite-volume homogeneous space.

Let $A$ be $d \times d$ diagonal matrices of det 1. $A$ preserves $N$, hence $\mathcal{S}$.

**Prop.** Let

$$\Omega = \left\{ \Lambda_0 \in G_0/\Gamma_0 : \Lambda_0 \text{ intersects } \bar{B}\left(0, \frac{\sqrt{d}}{2}\right) \right\}.$$

Then MC is equivalent to:

any $A$-orbit in $G_0/\Gamma_0$ intersects $\Omega$.

**Proof.** Move a vector $\vec{x}$ with $N(\vec{x}) = 2^{-d}$ using $A$ so all non-zero coordinates are equal. Norm $\leq \sqrt{\sum_1^d \left(\frac{1}{2}\right)^2} = \frac{\sqrt{d}}{2}$.

**Note** $\Omega$ has nonempty interior.

## A strategy (Remak-Davenport)

Let $G = \mathsf{SL}_d(\mathbb{R}),\ \Gamma = \mathsf{SL}_d(\mathbb{Z})$. Then $G/\Gamma$ is space of unimodular lattices. Have map $\pi : G_0/\Gamma_0 \to G/\Gamma$ with compact fiber – $\pi^{-1}(\Lambda) = \mathbb{R}^d/\Lambda$ (a torus).

Define

$$\mathrm{cov}(\Lambda) = \min\left\{ r : \mathbb{R}^d = \bigcup_{\lambda \in \Lambda} \bar{B}(\lambda, r) \right\},$$

and

$$\mathcal{C} = \left\{ \Lambda : \mathrm{cov}(\Lambda) \le \sqrt{d}/2 \right\}$$

**Note** $\pi^{-1}(\mathcal{C}) \subset \Omega$.

**Cor.** (MC) follows from: any $A$-orbit in $G/\Gamma$ intersects $\mathcal{C}$. C

**Thm. (Birch–Swinnerton-Dyer, 1956)** Enough to show any *bounded* orbit intersects $\mathcal{C}$.

**MacBeath's (sub-)strategy**

**Thm (MacBeath, 1961)**

(i) $N\Gamma \subset \mathcal{C}$.

(ii) $KN\Gamma \subset \mathcal{C}$.

(iii) $AKN\Gamma$ has non-empty interior.

**Remarks.**

1. Isomorphism $AKN\Gamma = DOTU$.

2. $AKN\Gamma = G/\Gamma$ for $d = 2, 3$.

3. $AKN\Gamma \neq G/\Gamma$ for $d = 64$ and for all $d \geq d_0$.

4. $KN\Gamma \subsetneq \mathcal{C}$.

## Well-rounded lattices

**Def.** $\Lambda$ is well-rounded if the shortest non-zero vectors in $\Lambda$ span $\mathbb{R}^d$.

**Thm. (McMullen '04).** The closure of any bounded $A$-orbit contains a well-rounded lattice.

Based on **Topological theorem:**

If $\mathbb{R}^{d-1}$ is covered by sets $U_1, U_2, \ldots,$ and each connected component of $U_{i_1} \cap \cdots \cap U_{i_k}$ is a uniformly bounded distance from a codim $k$ affine hyperplane, then there is a point covered $d$ times.

Now let $\{v_1, v_2, \ldots\}$ be the primitive vectors of $\Lambda$ and

$$U_i = \{a \in A : av_i \text{ shortest in } a\Lambda\}.$$

**Q.** Are well-rounded lattices in $\mathcal{C}$?

Yes for $d \leq 8$.

**Stable lattices**

For $k = 1, \ldots, d$, let $\alpha_k(\Lambda) =$
$\min \left\{ \mathrm{covol}(\Lambda_0)^{1/k} : \Lambda_0 \subset \Lambda, \ \mathrm{rank}(\Lambda_0) = k \right\}.$

**Def.** $\Lambda$ is called *stable* if $\alpha_k(\Lambda) \geq 1$ for $k = 1, \ldots, d$.

**Thm. (Shapira + W.)** The closure of any bounded $A$-orbit contains a stable lattice.

Relies on McMullen's topological theorem.

**Q.** Are stable lattices in $\mathcal{C}$? Yes, for $d \leq 6$.

**Cor 1.** To prove MC in dim $d$, enough to check the *local maxima of* cov. These are finitely many and under investigation (Dutour Sikiric et al).

**Cor 2.** For any $\Lambda$, there is $a \in A$ such that $a\Lambda$ has no vector of length less than 1.

This improves earlier bounds of Siegel and Davenport.

**Another result of Shapira**

**Thm.**

For any $\Lambda \in G/\Gamma$, a.e. $\Lambda_0 \in \pi^{-1}(\Lambda)$ satisfies MC.

**Idea of proof** If $\Lambda$ has a divergent $A$-orbit, then this orbit intersects $\mathcal{C}$ (MacBeath + Tomanov-W).

If there is $a_n \in A, a_n \to \infty$ such that $a_n\Lambda$ converges in $G/\Gamma$, can write

$a_n\Lambda = g_n\gamma_n\mathbb{Z}^d$ where $\gamma_n \in \Gamma$ and $g_n \to g_0$; action of $\gamma_n$ on fiber $\mathbb{R}^d/\mathbb{Z}^d$ is *mixing*.

MAZAL TOV URI AND YASMIN!!!