Lattice statistics in large dimensions

Anders Södergren

Department of Mathematics
Uppsala University

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Setting the stage

- Consider $X_n = \mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$ as a space of lattices:

$$L = \mathbb{Z}^n g \subset \mathbb{R}^n \longleftrightarrow \mathrm{SL}(n,\mathbb{Z})g \in X_n.$$

- Given $L \in X_n$, order the non-zero vector lengths in L as $0 < \ell_1 \leq \ell_2 \leq \ell_3 \leq \ldots$.
- For $j \geq 1$ and $t \geq 0$, define

$$\mathcal{V}_j(L) := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\ell_j^n,$$
$$\tilde{N}_t(L) := \#\{j : \mathcal{V}_j(L) \leq t\}.$$

- **Question**: Can we describe the behavior of the process $\{\tilde{N}_t(\cdot), t \geq 0\}$ as $n \to \infty$?

Theorem 1

$\{\tilde{N}_t(\cdot), t \geq 0\}$ *tends to a Poisson process on* $\mathbb{R}^+$ *with intensity* $\frac{1}{2}$.

In other words: As $n \to \infty$, $\mathcal{V}_1(L), \mathcal{V}_2(L), \ldots, \mathcal{V}_N(L)$ behave like the first N points of a Poisson process on $\mathbb{R}^+$ (any fixed N).

Theorem 1′

For a random flat torus $\mathbb{R}^n/\Lambda$ *with* $\Lambda \in X_n$ *the non-zero eigenvalues (properly normalized) behave like the points of a Poisson process on* $\mathbb{R}^+$ *as* $n \to \infty$.

Further results

- **Angles** between the N shortest lattice vectors.
  We prove: $\sqrt{n} \cdot (\frac{\pi}{2} - \text{"angle}_{ij}\text{"})$ tend to $\binom{N}{2}$ **independent gaussians** as $n \to \infty$.

- **Epstein zeta function**:

$$E(L, s) := {\sum_{\mathbf{m} \in L}}' |\mathbf{m}|^{-2s}$$

  We prove results on:
  - Value distribution at $s = cn$ as $n \to \infty$ ($c > \frac{1}{2}$ fixed).
  - Probability of $E(L, s)$ having a zero with $\text{Re}(s) > cn$ as $n \to \infty$.