Two minute horocycles

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Equidistribution of closed horocycles

Let $\mathbb{H} := \{x + \mathrm{i}y : x \in \mathbb{R},\ y \in \mathbb{R}_{>0}\}$ (the upper halfplane), $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ (the modular group), $X = \Gamma\backslash\mathbb{H}$ the modular surface.

For $f : [0,1] \times X \to \mathbb{R}$ bounded continuous,

$$\lim_{y\to 0} \int_0^1 f(x, x + \mathrm{i}y) = \int_0^1 \int_X f(\xi, x + \mathrm{i}y)\, d\xi\, \frac{dx\, dy}{y^2}.$$

There are three ways to prove this theorem: (1) exploit mixing, (2) harmonic analysis or (3) Ratner's classifation of measures invariant under unipotent flows.

See Zagier, Sarnak, Eskin & McMullen, Hejhal, Strömbergsson, . . .

Rational points on closed horocycles

For $f : [0,1] \times X \to \mathbb{R}$ bounded continuous,

$$\lim_{q\to\infty} \frac{1}{\varphi(q)} \sum_{\substack{p=1\\ \gcd(p,q)=1}}^{q-1} f\left(\frac{p}{q}, \frac{p}{q} + \mathrm{i}\frac{C}{q^2}\right) = \int_0^1 \int_0^1 f(\xi, x + \mathrm{i}C^{-1})\, d\xi\, dx$$

This follows from classical bounds on Kloosterman sums.

(A higher dimensional variant of this is used in my paper on Frobenius numbers.)

Irrational points...

Let $\alpha \notin \mathbb{Q}$.

I conjecture (but don't know how to prove) the following:

$$\lim_{q\to\infty} \frac{1}{\varphi(q)} \sum_{\substack{p=1 \\ \gcd(p,q)=1}}^{q-1} f\left(\frac{p}{q}, \alpha\frac{p}{q} + \mathrm{i}\frac{C}{q^2}\right) = \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} \int_0^1 f(\xi, x+\mathrm{i}y)\, d\xi\, \frac{dx\, dy}{y^2}$$

The last statement would imply (f bounded continuous)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(\frac{n}{N}, \alpha\frac{n}{N} + \mathrm{i}\frac{C}{N^2}\right) = \int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} \int_0^1 f(\xi, x+\mathrm{i}y)\, d\xi\, \frac{dx\, dy}{y^2}$$

If this holds also for f with $f(x+\mathrm{i}y) \sim y^{1/2}$ as $y \to \infty$, then the pair correlation of the sequence

$$\frac{\alpha}{N} n^2 \bmod 1, \qquad n = 1, \ldots, N$$

is Poission $\longrightarrow$ semiclassical limit of boxed oscillator & quantum maps (Zelditch CMP 1998); for αn^2 see Sinai (Phys D 1988), Rudnick & Sarnak (CMP 1998), JM & Strömbergsson (GAFA 2003), Heath-Brown (Proc Phil Soc Camb 2009)

Connection to quantitative Oppenheim (Eskin, Margulis & Mozes, Ann Math 1998 & 2005):

$$N\left(\frac{\alpha}{N} m^2 - \frac{\alpha}{N} n^2 + k\right) = \alpha(m^2 - n^2) + Nk$$

is a form of signature (2,2)—but here we FIX N and count values with integer $m, n, k \asymp N$.