Regularity of Diophantine Arnold tongues at the critical point

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EU-Young and mobile workshop: Dynamical systems and number theory

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Basic concepts

Given $f \in \mathrm{Diff}^0_+(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, we define the rotation number

$$\rho(f) = \lim_{|n|\to\infty} \frac{f^n(x_0) - x_0}{n} \pmod 1 \quad \forall x_0 \in \mathbb{T}.$$

Some properties:

- $\rho(f)$ exists for all x_0 and is independent of x_0.
- ρ is a topological invariant.
- ρ is continuous in the $\mathcal{C}^0$ topology.
- If $R_\theta(x) = x + \theta$, then $\rho(R_\theta) = \theta$.

A natural question in Dynamical Systems is wheter a particular circle map is conjugated to a rotation.

Theorem (Denjoy, 32)

If $f \in \mathrm{Diff}^{\omega}_{+}(\mathbb{T})$ with $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, then f is topologically conjugate to a rigid rotation $R_{\rho(f)}$.

More interesting is to ask about the regularity of this conjugation. It is well known that the answer depends on arithmetic properties of the rotation number.

Definition

We say that θ is Diophantine if there exist constants $C > 0$ and $\tau \geq 1$ such that $|\theta - l/k| \geq C|k|^{\tau}$, $\forall (l,k) \in \mathbb{Z} \times \mathbb{Z}_{*}$.

Theorem (Arnold 61, Herman 79, . . ., Khanin & Teplinsky 09)

If $f \in \mathrm{Diff}^{\omega}_{+}(\mathbb{T})$ and $\theta = \rho(f)$ is Diophantine, then f is analitically conjugate to a rigid rotation $R_{\rho(f)}$.

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The theory of smooth equivalence of critical maps is much less developed.

Definition

We denote $\mathfrak{C}^{2k+1}$, the space of critical circle maps of order $2k+1$, given by analytic functions f that are strictly increasing in $\mathbb{R}$ and satisfy

- $f(x+1) = f(x) + 1$.
- $f^{(j)}(0) = 0$ for all $0 < j \leq 2k$, and $f(0)f^{(2k+1)}(0) \neq 0$.

We will consider parametric families as follows:

$$f_{\omega,\varepsilon}(x) = x + \omega - \frac{\varepsilon}{2\pi} g(2\pi x), \qquad g \text{ periodic.}$$

such that

$$f_{\omega,\varepsilon} \in \mathrm{Diff}^{\omega}_{+}(\mathbb{T}) \quad \text{for} \quad \varepsilon < 1, \qquad f_{\omega,1} \in \mathfrak{C}^{2k+1}$$

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Definition

Arnold tongues are defined as the subsets

$$T_\theta = \{(\omega, \varepsilon) \,:\, \rho(\omega, \varepsilon) = \theta\}, \qquad \theta \in [0, 1)$$

Let us recall that

- If $\theta \in \mathbb{Q}$, then T_θ is a set with interior.
- If $\theta \notin \mathbb{Q}$, then T_θ is a continuous curve which is the graph of a function $\varepsilon \mapsto \alpha(\varepsilon)$, with $\alpha(0) = \theta$.
- If $\theta \in \mathcal{D}$, then $\varepsilon \mapsto \alpha(\varepsilon)$ is analytic (Risler, 99).

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We study numerically these sets using two efficient numerical methods.

- **Averaging-extrapolation method**: high precision with few iterates.

[SV06] T. M. Seara and J. Villanueva. *On the Numerical Computation of Diophantine Rotation Numbers of Analytic Circle Maps.* Physica D 217 (2006) pp. 107-120.

[LV08] A. Luque and J. Villanueva. *Computation of derivatives of the rotation number for parametric families of circle diffeomorphisms. Physica D 237 (2008) pp. 2599-2615.*

- **Parameterization method**: it allows to compute r-Sobolev norms of the conjugacy at every point of the Tongue.

Method 1: averaging-extrapolation

Introduce the recurrent sums

$$D_\mu^d S_N^0(f) := D_\mu^d(f^N(x_0) - x_0), \qquad D_\mu^d S_N^p(f) := \sum_{j=1}^{N} D_\mu^d S_j^{p-1}(f)$$

Proposition

Let us consider $f \in \mathrm{Diff}_+^\omega(\mathbb{T})$ such that $\theta = \rho(f) \in \mathcal{D}$ and $D_\mu^d \theta$ exist. Then the averages of the above sums satisfy the expression

$$D_\mu^d \tilde{S}_N^p(f) = \binom{N+p}{p+1}^{-1} S_N^p(f) = D_\mu^d \theta + \sum_{l=1}^{p-d} \frac{D_\mu^d A_l^p}{N^l} + D_\mu^d E^p(N)$$

where $D_\mu^d A_l^p$ are independent of N and $D_\mu^d E^p(N)$ is uniformly bounded by $\mathcal{O}(1/N^{p+1-d})$ if $D_\mu^d \theta \neq 0$.

Proofs in [SV06] for $d = 0$ and [LV08] for $d > 0$.

Method 2: parametrization method

Assume that for any $\varepsilon < 1$, we have an approximation of the Arnold tongue, i.e. ω and h such that

$$f_\omega(h(x)) = h(x+\theta) + e(x),$$

where $e : \mathbb{T} \to \mathbb{T}$. To implement a Newton method, we consider corrections $\bar{\omega} = \omega + \Delta_\omega$ and $\bar{h} = h + \Delta_h$ which are given by solving (at least approximately) the following linearized equation

$$f'_\omega(h(x))\Delta_h(x) - \Delta_h(x+\theta) + \partial_\omega f_\omega(h(x))\Delta_\omega = -e(x).$$

The trick is to write $\Delta_h(x) = h'(x)\varphi(x)$ (following Moser 66) and we obtain the cohomological equation:

$$\varphi(x) - \varphi(x+\theta) = v(x), \qquad v(x) := -\frac{\partial_\omega f_\omega(h(x))\Delta_\omega + e(x)}{h'(x+\theta)}.$$

This equation is studied using Fourier series for periodic functions

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k \mathrm{e}^{2\pi \mathrm{i} k x},$$

where we denote $[f]_{\mathbb{T}} = \hat{f}_0$ the average of f. Thus, we obtain

$$\Delta_\omega = -\frac{[e]_{\mathbb{T}}}{[\partial_\omega f_\omega \circ h]_{\mathbb{T}}}, \qquad \hat{\varphi}_k = \frac{\hat{v}_k}{1-\mathrm{e}^{2\pi \mathrm{i} k\theta}}.$$

Some motivating numerical experiments

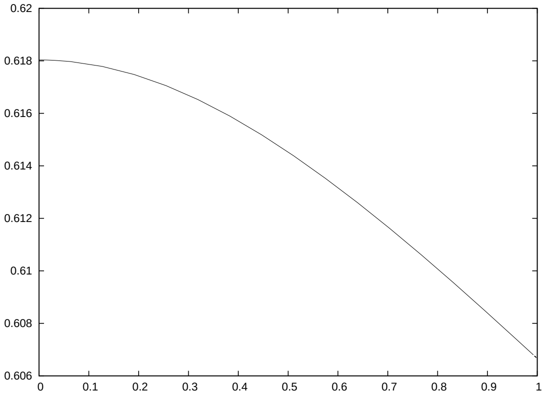


Figure: Graph $\varepsilon \mapsto \omega(\varepsilon)$ corresponding to $\theta = (\sqrt{5} - 1)/2$.

Taking derivatives in the expression

$$\rho(\varepsilon, \omega(\varepsilon)) = \theta,$$

we obtain

$$\omega'(\varepsilon) = -\frac{D_\varepsilon \rho(\omega(\varepsilon), \varepsilon)}{D_\omega \rho(\omega(\varepsilon), \varepsilon)}$$

and

$$\omega''(\varepsilon) = \frac{-(D_{\omega\omega}\rho(\omega(\varepsilon), \varepsilon)\omega'(\varepsilon) + 2D_{\omega\varepsilon}\rho(\omega(\varepsilon), \varepsilon))\omega'(\varepsilon) - D_{\varepsilon\varepsilon}\rho(\omega(\varepsilon), \varepsilon)}{D_\omega \rho(\omega(\varepsilon), \varepsilon)}.$$

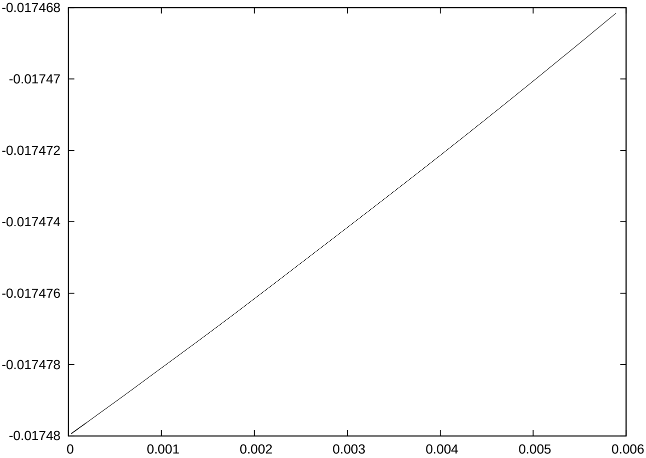


Figure: Graph of the derivative $(1-\varepsilon) \mapsto \omega'(\varepsilon)$.

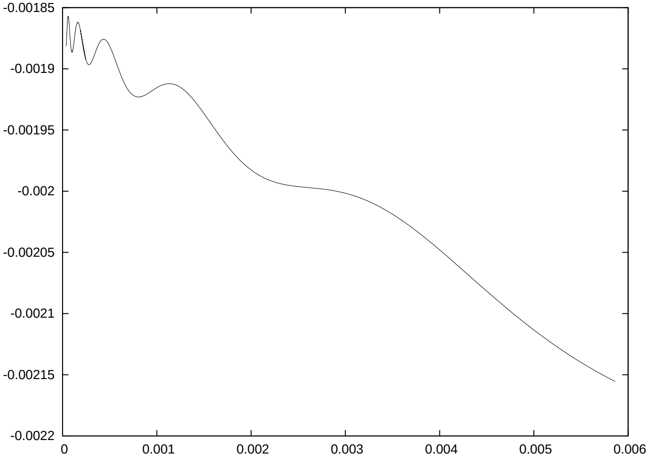


Figure: Graph of the derivative $(1-\varepsilon) \mapsto \omega''(\varepsilon)$.

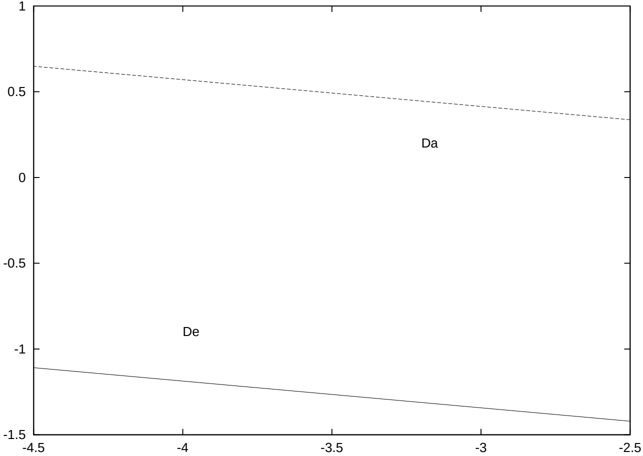


Figure: Graph of the derivatives $\log_{10}(1-\varepsilon) \mapsto \log_{10} D_\omega \rho(\omega(\varepsilon), \varepsilon)$ (upper graph) and $\log_{10}(1-\varepsilon) \mapsto \log_{10} D_\varepsilon \rho(\omega(\varepsilon), \varepsilon)$ (lower graph).

We fit the blow-up of the derivatives using the following expression

$$D_\omega \rho(\omega(\varepsilon), \varepsilon) \simeq \frac{0.884}{(1-\varepsilon)^{0.155}}, \qquad \varepsilon \simeq 1$$

$$D_\varepsilon \rho(\omega(\varepsilon), \varepsilon) \simeq \frac{0.015}{(1-\varepsilon)^{0.155}}, \qquad \varepsilon \simeq 1.$$

Next, we show more Arnold tongues (close to the critical point) for several families (of different criticallity) and rotation numbers of the form $\theta = [0, a, b, a, b, \ldots]$, where

$$\begin{aligned} \theta &= [a_0, a_1, \ldots, a_N, \ldots] \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_N + \ldots}}}} \end{aligned}$$

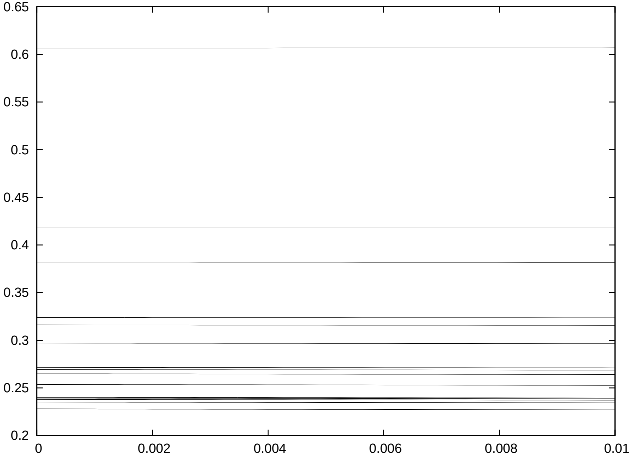


Figure: Cubic family $f^A_{\omega,\varepsilon}(x) = x + \omega - \dfrac{\varepsilon}{2\pi}\sin(2\pi x)$. .

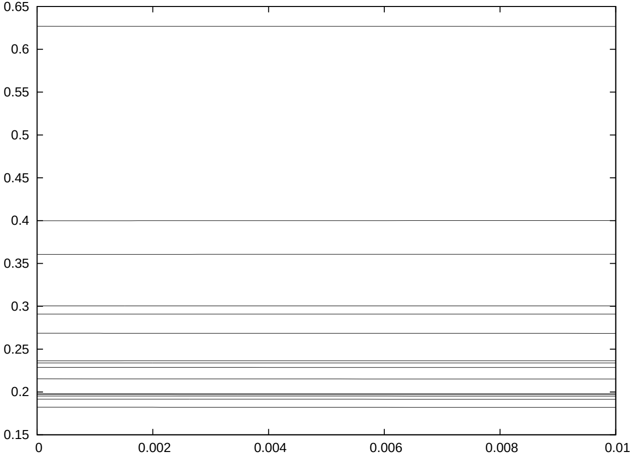
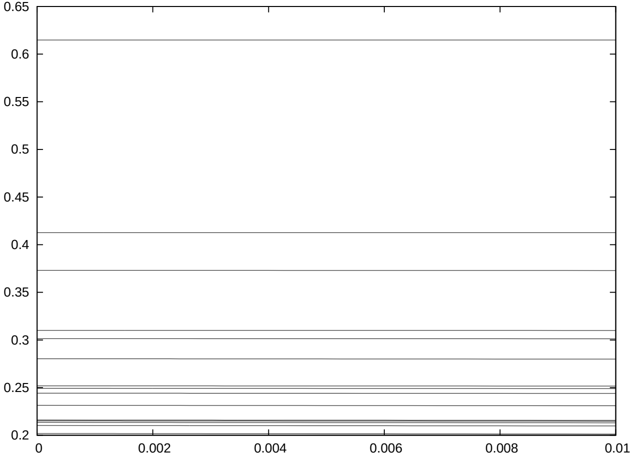


Figure: Cubic family $f^{C}_{\omega,\varepsilon}(x) = x + \omega - \dfrac{\varepsilon}{2\pi}\left(\kappa\sin(2\pi x) + \dfrac{1-\kappa}{2}\sin(4\pi x)\right)$ with $\kappa = 0.3$.



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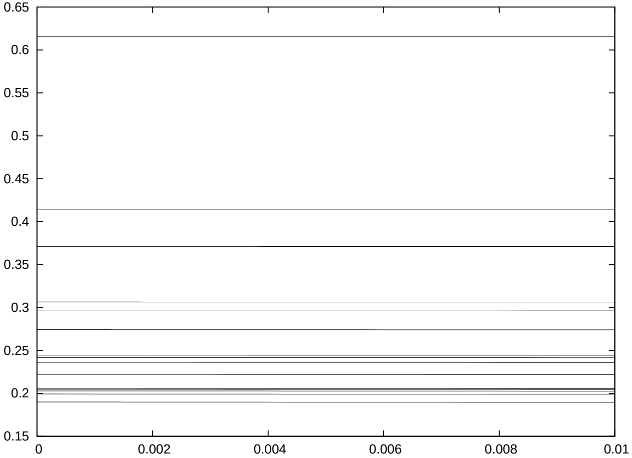


Figure: Cubic family $f^H_{\omega,\varepsilon}(x) = x + \omega - \dfrac{\varepsilon}{2\pi}\dfrac{(1-\kappa)\sin(2\pi x)}{1-\kappa\cos(2\pi x)}$ with $\kappa = 0.5$.

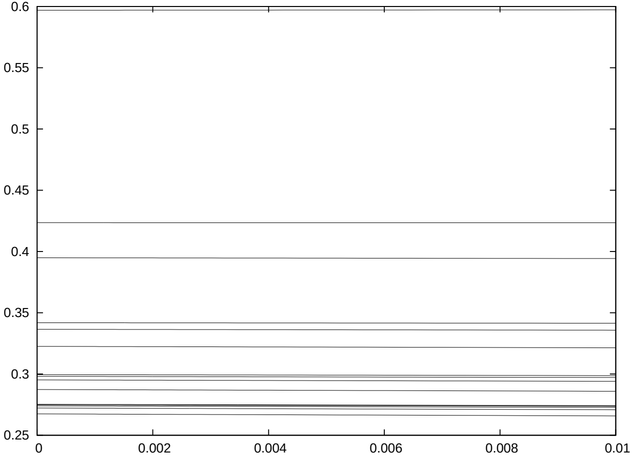


Figure: Quintic family $f^{C}_{\omega,\varepsilon}(x) = x + \omega - \frac{\varepsilon}{2\pi}\left(\kappa\sin(2\pi x) + \frac{1-\kappa}{2}\sin(4\pi x)\right)$ with $\kappa = 4/3$.

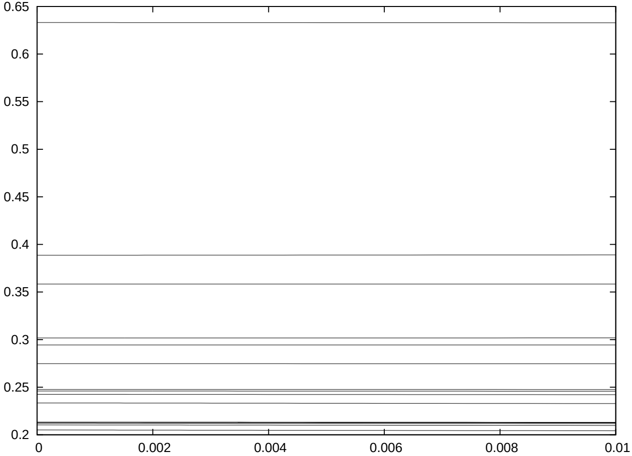
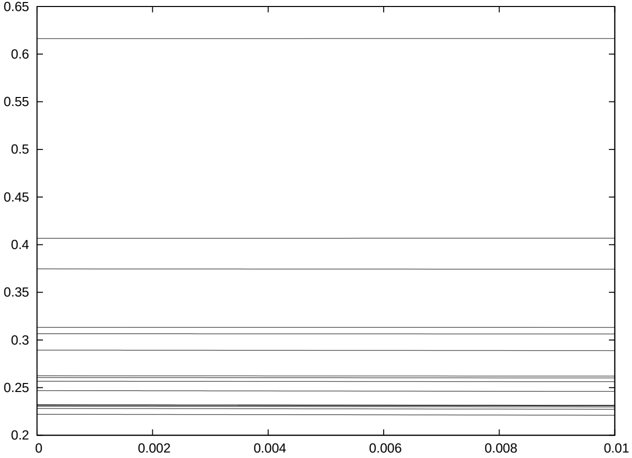


Figure: Quintic family ($\kappa = 0.6$)

$$f^{Q}_{\omega,\varepsilon}(x) = x + \omega - \frac{\varepsilon}{2\pi}\left(\kappa \sin(2\pi x) + \frac{9-8\kappa}{10}\sin(4\pi x) + \frac{3\kappa-4}{15}\sin(6\pi x)\right) .$$



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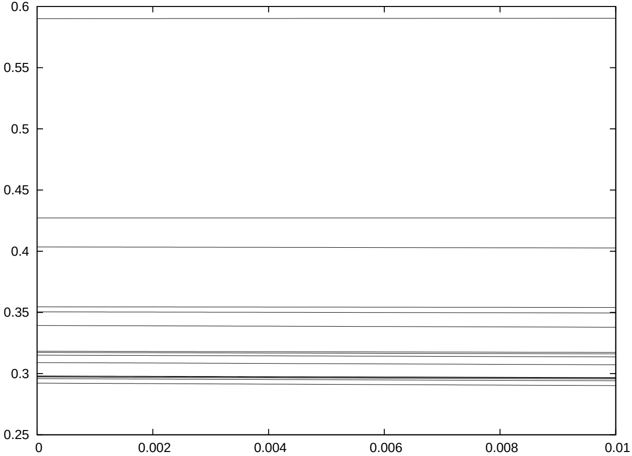

Figure: Septic family ($\kappa = 1.5$)

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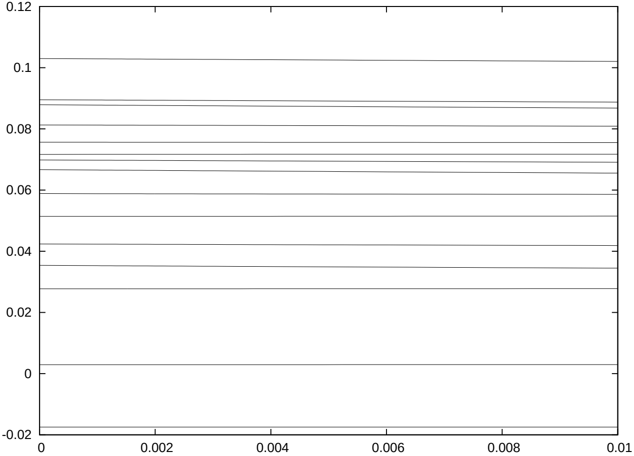
Next, we show the derivatives of the previous curves...
(they are quite nice, aren't they?)



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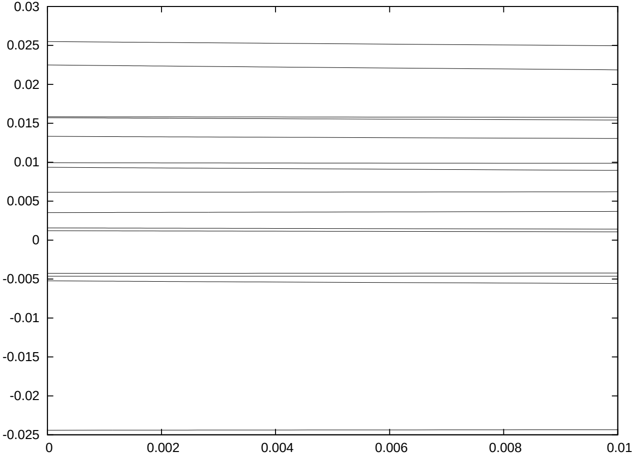

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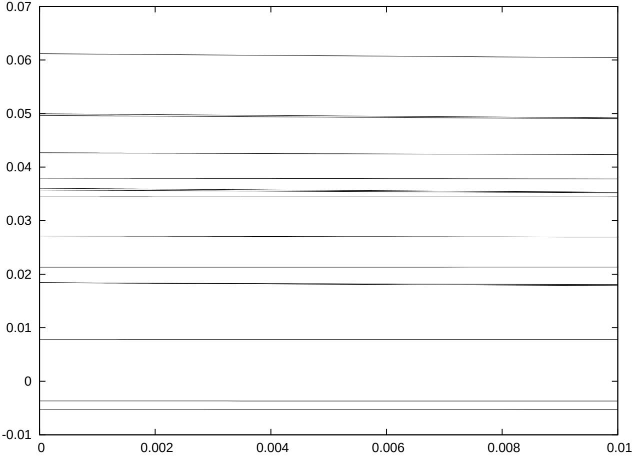

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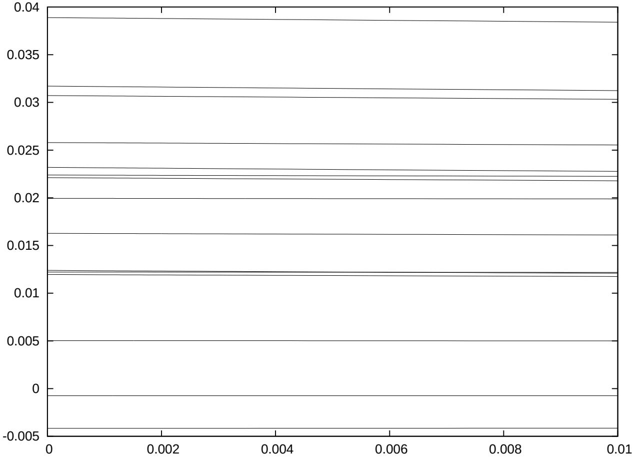

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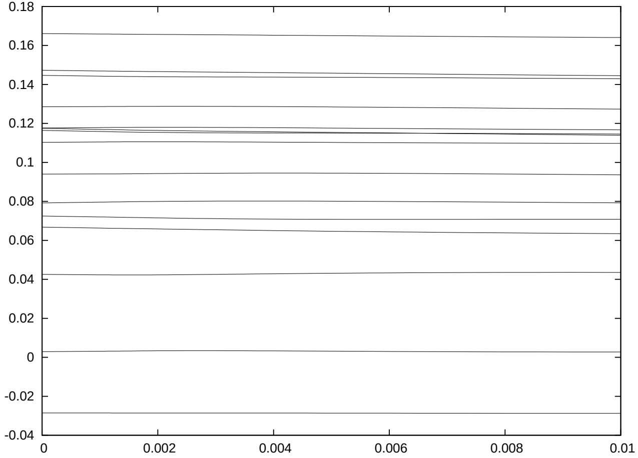

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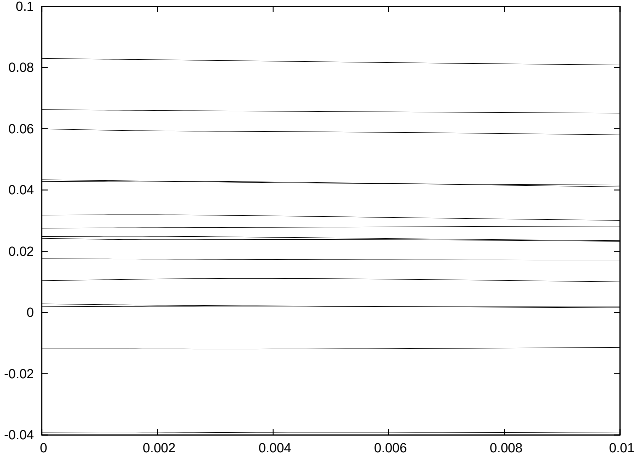

Figure: Quintic family ($\kappa = 0.6$)

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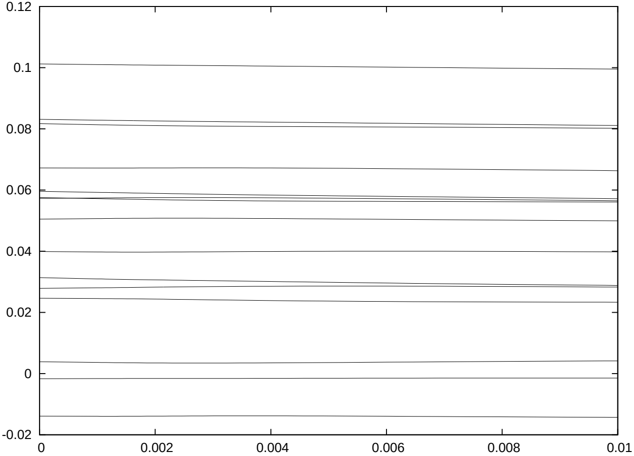

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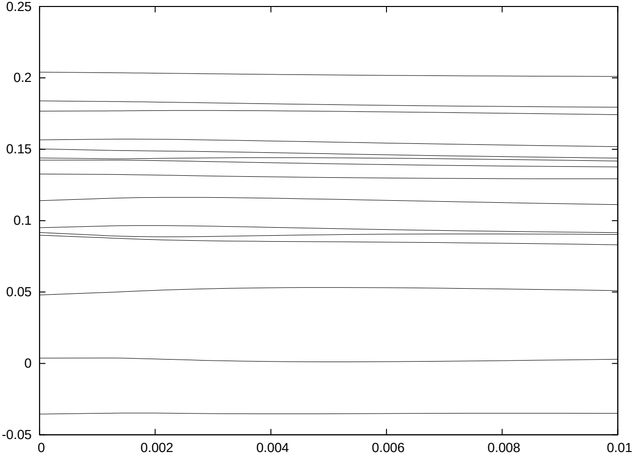

Figure: Septic family ($\kappa = 1.5$)

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We know that the derivatives of the rotation number blow-up by means of the law

$$D_\omega \rho(\omega(\varepsilon), \varepsilon) \simeq \frac{A}{(1-\varepsilon)^{B_*}} \text{ when } \quad \varepsilon \simeq 1$$

where B_* depends only on the order of criticallity (not on the map) and the rotation number.

In order to see the dependence on the rotation number we use the Bruno function $\mathcal{B}(\theta)$, which measures (roughly speaking) if θ is a good Diophantine number.

Cubic families.

θ	Arnold	$\kappa = 0.3$	$\kappa = 0.7$	$\kappa = 1/2$
$[0; 1, 1, 1, 1, \ldots]$	0.1552	0.1556	0.1550	0.1557
$[0; 2, 1, 2, 1, \ldots]$	0.1749	0.1747	0.1748	0.1739
$[0; 2, 2, 2, 2, \ldots]$	0.1660	0.1645	0.1659	0.1665
$[0; 3, 1, 3, 1, \ldots]$	0.2060	0.2038	0.2057	0.2048
$[0; 3, 2, 3, 2, \ldots]$	0.1811	0.1788	0.1806	0.1795
$[0; 3, 3, 3, 3, \ldots]$	0.1871	0.1849	0.1865	0.1858
$[0; 4, 1, 4, 1, \ldots]$	0.2382	0.2412	0.2385	0.2386
$[0; 4, 2, 4, 2, \ldots]$	0.2021	0.2061	0.2019	0.2015
$[0; 4, 3, 4, 3, \ldots]$	0.2035	0.2072	0.2031	0.2021
$[0; 4, 4, 4, 4, \ldots]$	0.2151	0.2182	0.2142	0.2136
$[0; 5, 1, 5, 1, \ldots]$	0.2702	0.2694	0.2700	0.2708
$[0; 5, 2, 5, 2, \ldots]$	0.2259	0.2250	0.2255	0.2248
$[0; 5, 3, 5, 3, \ldots]$	0.2226	0.2244	0.2237	0.2241
$[0; 5, 4, 5, 4, \ldots]$	0.2299	0.2320	0.2306	0.2319
$[0; 5, 5, 5, 5, \ldots]$	0.2465	0.2458	0.2461	0.2455

Quintic and Septic families.

θ	Arnold	$\kappa = 0.3$	$\kappa = 0.7$	$\kappa = 1/2$
$[0; 1, 1, 1, 1, \ldots]$	0.2124	0.2112	0.2114	0.2446
$[0; 2, 1, 2, 1, \ldots]$	0.2330	0.2395	0.2323	0.2687
$[0; 2, 2, 2, 2, \ldots]$	0.2257	0.2260	0.2283	0.2560
$[0; 3, 1, 3, 1, \ldots]$	0.2683	0.2642	0.2670	0.2958
$[0; 3, 2, 3, 2, \ldots]$	0.2411	0.2406	0.2430	0.2602
$[0; 3, 3, 3, 3, \ldots]$	0.2491	0.2482	0.2476	0.2874
$[0; 4, 1, 4, 1, \ldots]$	0.2982	0.2814	0.2985	0.3705
$[0; 4, 2, 4, 2, \ldots]$	0.2506	0.2490	0.2561	0.2829
$[0; 4, 3, 4, 3, \ldots]$	0.2596	0.2508	0.2485	0.3167
$[0; 4, 4, 4, 4, \ldots]$	0.2831	0.2542	0.2761	0.3132
$[0; 5, 1, 5, 1, \ldots]$	0.3256	0.3371	0.3293	0.3753
$[0; 5, 2, 5, 2, \ldots]$	0.2773	0.2851	0.2794	0.3178
$[0; 5, 3, 5, 3, \ldots]$	0.2844	0.2828	0.2741	0.3293
$[0; 5, 4, 5, 4, \ldots]$	0.3192	0.3206	0.3177	0.3471
$[0; 5, 5, 5, 5, \ldots]$	0.3167	0.3273	0.3121	0.3510

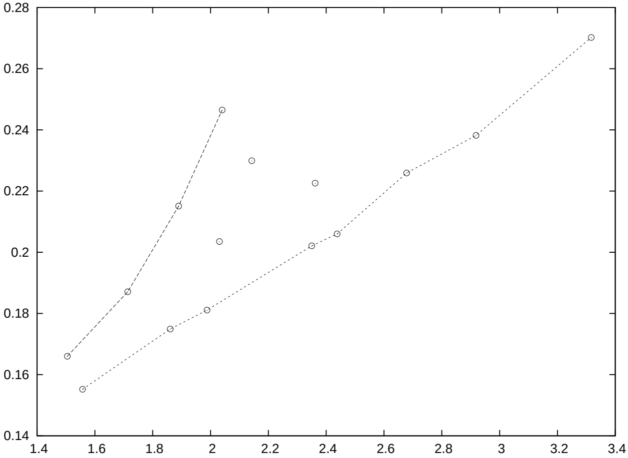


Figure: Cubic family $f^{A}_{\omega,\varepsilon}(x) = x + \omega - \frac{\varepsilon}{2\pi}\sin(2\pi x)$. .

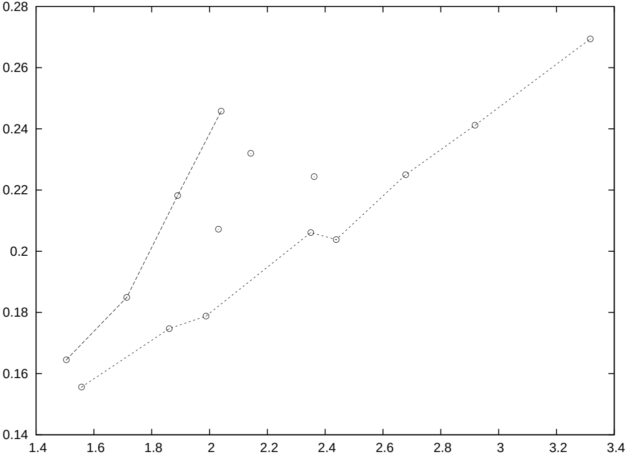


Figure: Cubic family $f^{C}_{\omega,\varepsilon}(x) = x + \omega - \frac{\varepsilon}{2\pi}\left(\kappa \sin(2\pi x) + \frac{1-\kappa}{2}\sin(4\pi x)\right)$ with $\kappa = 0.3$.

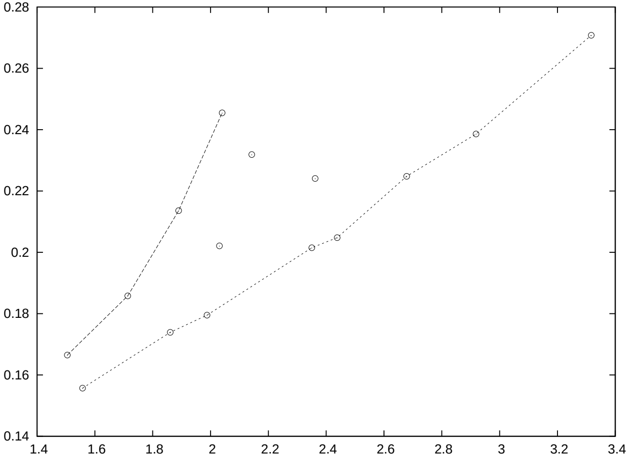


Figure: Cubic family $f^H_{\omega,\varepsilon}(x) = x + \omega - \dfrac{\varepsilon}{2\pi}\,\dfrac{(1-\kappa)\sin(2\pi x)}{1-\kappa\cos(2\pi x)}$ with $\kappa = 0.5$.

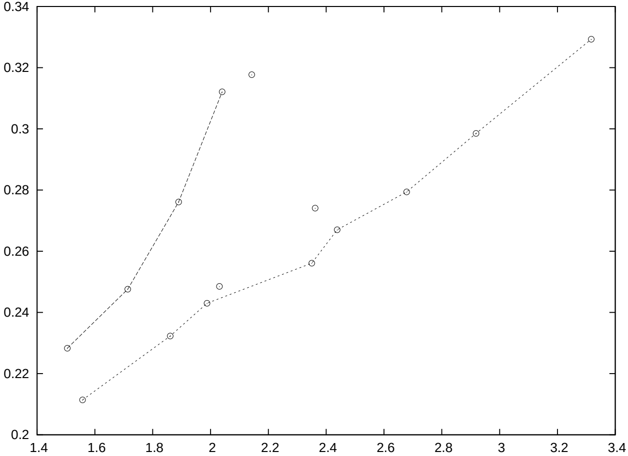


Figure: Quintic family ($\kappa = 0.9$)

$$f^{Q}_{\omega,\varepsilon}(x) = x + \omega - \frac{\varepsilon}{2\pi}\left(\kappa \sin(2\pi x) + \frac{9-8\kappa}{10}\sin(4\pi x) + \frac{3\kappa-4}{15}\sin(6\pi x))\right) .$$

Explanation in terms of Renormalization Group

- Renormalization Group (RG) provides powerful tools for the study of long term dynamics by the fact that “highly iterated maps, when observed in small scales, have forms that are largely independent of the map”.
- RG was introduced in Dynamical Systems by Feigenbaum (period-doubling universality in unimodal maps).
- Extensions for circle maps are found in the work (experimental and rigorous) of de Faria, Feigenbaum, Kadanoff, Lanford, Mestel, Östlund, Rand, Sethna, Shenker, Siggia, etc.
- Unfortunately, we do not have enough time to introduce the RG in this context, but we will comment some relevant properties.

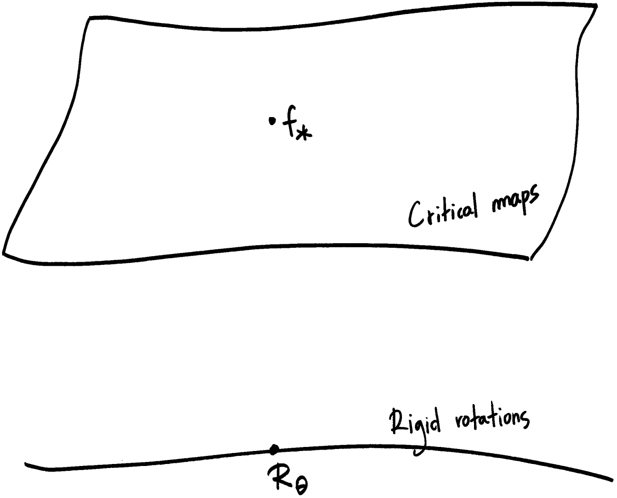
- Given $\theta \in [0, 1)$, RG acts in the space of circle maps in such a way that the spaces

  $$\{f \in \mathrm{Diff}^{\omega}_{+}(\mathbb{T}) \,:\, \rho(f) = \theta\} \qquad \{f \in \mathfrak{C}^3 \,:\, \rho(f) = \theta\}.$$

  are invariant under RG.

- We can think of $\mathfrak{C}^3$ as a cell of lower codimension attached to the boundary of $\mathrm{Diff}^{\omega}_{+}(\mathbb{T})$ (as in the families that we considered before).

- It turns out that RG has two fixed points in the space of circle maps.

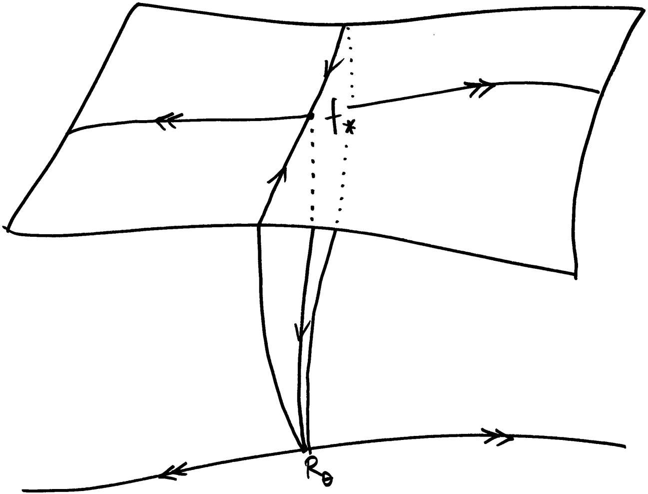

f_*
Critical maps
Rigid rotations
R_θ

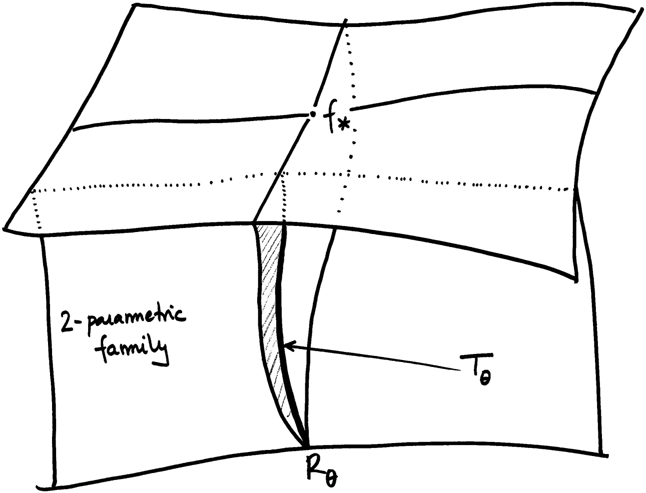
- The trivial fixed point R_θ has a one-dimensional unstable manifold $W^u_{R_\theta}$ given by the curve of rigid rotations. Moreover, $W^s_{R_\theta}$ has codimension one and consists in maps of rotation number θ.

- The non-trivial fixed point f_* has a two dimensional unstable manifold $W^u_{f_*}$ that contains the curve of rigid rotations in its closure. Moreover, $W^s_{f_*}$ has codimension two and consists in maps of rotation number θ.

- The eigenvalue of $W^u_{f_*}$ restricted to $T\mathfrak{C}^3$ is $\delta = 2.83362\ldots$. The associate direction is transverse to the maps of rotation number θ.

- The remaining direction of $W^u_{f_*}$ is transverse to $\mathfrak{C}^3$ and has eigenvalue $\gamma = 1.6604\ldots$


f_*
R_θ


f_*
2-parametric family
T_θ
R_θ

Conclusions

- Then, it turns out that the Arnold tongue of rotation number θ corresponds to the intersection of a two-parametric family of maps with a Normally Hyperbolic Invariant Manifold. From Fenichel's theory, the regularity is bounded by

$$r \leq \frac{\log \delta}{\log \gamma}.$$

- Furthermore, the RG explains the universal properties observed numerically. In particular that the behaviour in the critical point depends only on the order of criticallity and the rotation number.
- RG also provides assymtotic formulas for the exponents B_*.