Sums over fractional parts of $n\alpha - \gamma$

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Definition and previous work

- Definition: $N \in \mathbb{N}, \gamma, \alpha \in \mathbb{R}$,

$$S_N(\alpha, \gamma) = \sum_{n=1}^{N} \frac{1}{n\|n\alpha - \gamma\|}.$$

- Theorem (Schmidt, 1964): For any $\gamma \in \mathbb{R}, \epsilon > 0$,

$$\log^2 N \ll S_N(\alpha, \gamma) \ll \log^{2+\epsilon} N,$$

for a.e. $\alpha \in \mathbb{R}$.

Subsequent improvements

- Theorem (Kuipers and Niederreiter, 1974): If $\psi : \mathbb{N} \to \mathbb{R}$ is nonnegative and nondecreasing and if

$$\inf_{n\in\mathbb{N}} n\psi(n)\|n\alpha\| > 0$$

then

$$S_N(\alpha, 0) \ll \log^2 N + \psi(N) + \sum_{1\leq n\leq N} \frac{\psi(n)}{n}.$$

- Corollary: If α is of bounded type then

$$S_N(\alpha, 0) \ll \log^2 N.$$

Our result

- Theorem (Victor, A., Sanju): For any $\gamma \in \mathbb{R}$ and for almost every $\alpha \in \mathbb{R}$,
$$S_N(\alpha, \gamma) \ll \log^2 N.$$
- Potential applications: metric number theory, estimates for exponential sums.