# On uniform Diophantine exponents

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Consider a system of linear equations

$$\Theta \mathbf{x} = \mathbf{y}$$

with $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ and $\Theta$ an $n \times m$ real matrix.

### Definition

The supremum of real numbers $\gamma$, such that there are infinitely many $\mathbf{x} \in \mathbb{Z}^m$, $\mathbf{y} \in \mathbb{Z}^n$ satisfying the inequality

$$|\Theta \mathbf{x} - \mathbf{y}|_\infty \leqslant |\mathbf{x}|_\infty^{-\gamma},$$

is called the *individual Diophantine exponent* of $\Theta$ and is denoted by $\beta(\Theta)$.

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The supremum of real numbers $\gamma$, such that for each $t$ large enough there are $\mathbf{x} \in \mathbb{Z}^m$, $\mathbf{y} \in \mathbb{Z}^n$ satisfying the inequalities

$$0 < |\mathbf{x}|_\infty \leqslant t, \qquad |\Theta \mathbf{x} - \mathbf{y}|_\infty \leqslant t^{-\gamma},$$

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# Inequalities by Jarník and Apfelbeck

**Theorem (Jarník)**

*If $n=1$, $m=2$, then*

$$\alpha(\Theta)^{-1}+\alpha(\Theta^{\intercal})=1.$$

**Theorem (Apfelbeck)**

$(i)$ *We always have*

$$\alpha(\Theta^{\intercal})\geqslant\frac{n\alpha(\Theta)+n-1}{(m-1)\alpha(\Theta)+m}.$$

$(ii)$ *If $m>1$ and $\alpha(\Theta)>(2(m+n-1)(m+n-3)+m)/n$, then*

$$\alpha(\Theta^{\intercal})\geqslant\frac{1}{m}\left(n+\frac{n(n\alpha(\Theta)-m)-2n(m+n-3)}{(m-1)(n\alpha(\Theta)-m)+m-(m-2)(m+n-3)}\right).$$

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# New inequalities

## Theorem 1

*For all positive integers n, m, not equal simultaneously to* 1*, we have*

$$\alpha(\Theta^{\intercal}) \geqslant \begin{cases} \dfrac{n-1}{m-\alpha(\Theta)}, & \text{if } \alpha(\Theta) \leqslant 1, \\ \dfrac{n-\alpha(\Theta)^{-1}}{m-1}, & \text{if } \alpha(\Theta) \geqslant 1. \end{cases}$$

# Inequalities by Khintchine, Laurent, Bugeaud

### Theorem (Khintchine)

*If $n=1$, then*

$$\frac{\beta(\Theta)}{(m-1)\beta(\Theta)+m}\leqslant\beta(\Theta^{\intercal})\leqslant\frac{\beta(\Theta)-m+1}{m}.$$

### Theorem (Laurent, Bugeaud)

*If $n=1$, then*

$$\frac{(\alpha(\Theta)-1)\beta(\Theta)}{((m-2)\alpha(\Theta)+1)\beta(\Theta)+(m-1)\alpha(\Theta)}\leqslant\beta(\Theta^{\intercal})\leqslant$$

$$\leqslant\frac{(1-\alpha(\Theta^{\intercal}))\beta(\Theta)-m+2-\alpha(\Theta^{\intercal})}{m-1}.$$

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# Dyson's inequality

## Theorem (Dyson)

*For all n, m, not equal simultaneously to* 1*,*

$$\beta(\Theta^{\intercal}) \geqslant \frac{n\beta(\Theta)+n-1}{(m-1)\beta(\Theta)+m}.$$

# New inequalities

## Theorem 2

*For all positive integers n, m, not equal simultaneously to* 1*, we have three inequalities*

$$\beta(\Theta^{\intercal}) \geqslant \frac{n\beta(\Theta) + n - 1}{(m-1)\beta(\Theta) + m},$$

$$\beta(\Theta^{\intercal}) \geqslant \frac{(n-1)(1+\beta(\Theta)) - (1-\alpha(\Theta))}{(m-1)(1+\beta(\Theta)) + (1-\alpha(\Theta))},$$

$$\beta(\Theta^{\intercal}) \geqslant \frac{(n-1)(1+\beta(\Theta)^{-1}) - (\alpha(\Theta)^{-1} - 1)}{(m-1)(1+\beta(\Theta)^{-1}) + (\alpha(\Theta)^{-1} - 1)}.$$

# *The End*