Computational Tools for Combinatorial and Geometric Group Theory

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Standard Problems:

1. Is the group trivial?

2. Is the group infinite?

3. Are these two fp-groups isomorphic?

4. Is the group nilpotent, soluble, ...

5. Is the word problem for the group soluble?

The aim of this talk is to review the current set of tools for working with general fp-groups.
Contents

1. Determining automatic structure.
2. Families of groups with solvable word problems.
3. Constructing quotients of fp-groups.
4. Constructing subgroups of finite (and small) index for fp-groups.
5. A procedure for proving that certain fp-groups are infinite.
Automatic Groups

Let $G$ be a group generated as a monoid by the set $A$. Then $G$ is said to be \textit{automatic} wrt $A$ if there exists:

- A FSA $W$ with input alphabet $A$ such that, for each $g \in G$, $W$ accepts a word of $A^*$ representing it.

- For each $x \in A \cup \{\epsilon\}$, a 2-variable FSA $M_x$, such that for $v, w \in A^*$, $(v, w)^+ \in L(M_x)$ iff $v, w \in L(W)$ and $vx =_G w$.

$W$ is called the \textit{word acceptor} automaton for $G$.

The $M_x$ are called the \textit{multiplier} automata for $G$.

Collectively, $W$ and the $M_x$ are called the \textit{automatic structure} for $G$ wrt $X$. 
A is assumed to be $X \cup X^{-1}$ for generating set $X$ of $G$.

Assume the elements of $A$ are well-ordered. We can then define the *shortlex* ordering of $A^*$.

If the automatic structure for $G$ has the property that $W$ accepts precisely the minimal words for the group elements under some shortlex ordering of $A^*$ then it is called a *shortlex automatic structure* for $G$.

In that case $G$ is said to be *shortlex automatic* wrt $X$.

See Chapter 13 of *Handbook of Computational Group Theory* by Holt, Eick and O’Brien.
Automatic Groups (cont)

Having the automatic structure for an fp-group $G$ makes the following calculations possible:-

1. Determine if the group $G$ is finite or infinite.
2. Compute the growth function for $G$.
3. Enumerate the elements of $G$ without repetition.
4. Calculate a normal form for any word in $X^*$.
5. Perform arithmetic with the elements of $G$.
6. Test membership of quasiconvex subgroups of $G$. 
Thurston’s geometrization conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that have one of eight types of geometric structure.

A hyperbolic 3-manifold is a 3-manifold with a complete Riemannian metric of constant sectional curvature $-1$.

Any closed, irreducible, atoroidal 3-manifold with infinite fundamental group is hyperbolic.
Hyperbolic 3-Manifolds

Of the 8 types of geometry arising in the classification of 3-manifolds 7 are well-understood.

Thus, there is great interest in understanding the structures of hyperbolic 3-manifolds with the hope of discovering some underlying organization which could lead to a classification.

In the 1980s Bill Thurston and Jim Cannon recognized the connection between hyperbolic fundamental groups and automata.

Not long afterwards Derek Holt and Bob Gilman (separately) showed that the Knuth-Bendix procedure could be used to construct the automatic structure for an fp-group.
Finding the Automatic Structure

- The Knuth-Bendix method has been implemented for groups by Derek Holt and Charlie Sims.
- Epstein, Holt and Rees implemented a first program for constructing automatic structures in the late 1980s.
- Subsequently, Holt implemented an improved version which forms part of his KBMAG package.
- A version of KBMAG was installed in Magma in the early 2000s.
Finding the Automatic Structure

It was hoped that having a program able to solve the word problem for the fundamental groups would enable their properties to be studied in detail.

Early in 1990s Hodgson and Weeks compiled a census of 10,986 small-volume closed hyperbolic 3-manifolds. However, it seems that attempts to obtain automatic structures of these manifolds generally failed. Automatic structures are hard to compute.

In 2010 I decided to attempt to compute the automatic structures for a large number of these 3-manifolds (Derek was co-opted).

For each group processed, the automatic structure was saved and can be read back into Magma.
Hyperbolic Manifold $m003(-4, 3)$

> DB := Database( "H3Manifold" );
> R := Manifold(DB, "m003(-4,3)" );
> R;

rec< ... |
  Name := m003(-4,3),
  Volume := 1.2637089999999999999999999999997,
  Homology := [ 5, 5 ],
  KnownPosBettiCover := true,
  GoodCover := [ * PSL2, 16 * ],
  Group := Finitely presented group on 2 generators
    Relations
    x^2 * y * x^-1 * y^3 * x^-1 * y = 1
    x * y * x^2 * y^-2 * x^2 * y = 1,
]>
Automatic Structure for $m003(-4, 3)$

> SetVerbose("KBMAG",1);
> G := R`Group;
> time A := AutomaticGroup(G);

#Halting with 33765 equations.
#First word-difference machine with 121 states computed.
#Second word-difference machine with 123 states computed.
#System is confluent, or halting factor condition holds.
#Word-acceptor with 1053 states computed.
#General multiplier with 8529 states computed.
#Validity test on general multiplier succeeded.
#General length-2 multiplier with 8641 states computed.

#Checking inverse and short relations.
#Checking relation: $1^2*3*2*3 = 4*1*4^2$
#Checking relation: $1*3*1^2*4 = 4*2^2*3$
#Axiom checking succeeded.

Time: 9.090
The automatic group version $A$ of the fundamental group $G$ of $m003(-4,3)$ is loaded from the Magma database:

```plaintext
> DB := Database("H3Manifold");
> A := AutomaticGroup(DB, "m003(-4,3)");
> A;

An automatic group.
Generator Ordering = [ $.1, $.1^-1, $.2, $.2^-1 ]
Order = Infinity
The second word difference machine has 125 states.
The word acceptor has 1053 states.
```
Growth Function of m003(−4, 3)

Compute the growth function of $A$ as a rational function over $\mathbb{Z}$:

$$\text{gf} := \text{GrowthFunction}(A); \text{gf;}$$

$$( -t^{37} - 2*t^{35} - 12*t^{33} + 17*t^{32} - 36*t^{31} + 59*t^{30} - 72*t^{29} + 109*t^{28} - 111*t^{27} + 161*t^{26} - 150*t^{25} + 196*t^{24} - 141*t^{23} + 183*t^{22} - 78*t^{21} + 97*t^{20} - 3*t^{19} - 3*t^{18} + 97*t^{17} - 78*t^{16} + 183*t^{15} - 141*t^{14} + 196*t^{13} - 150*t^{12} + 161*t^{11} - 111*t^{10} + 109*t^{9} - 72*t^{8} + 59*t^{7} - 36*t^{6} + 17*t^{5} - 12*t^{4} - 2*t^{2} - 1 ) / ( t^{37} - 4*t^{36} + 6*t^{35} - 12*t^{34} + 24*t^{33} - 41*t^{32} + 56*t^{31} - 83*t^{30} + 136*t^{29} - 169*t^{28} + 227*t^{27} - 325*t^{26} + 402*t^{25} - 470*t^{24} + 593*t^{23} - 671*t^{22} + 722*t^{21} - 789*t^{20} + 809*t^{19} - 809*t^{18} + 789*t^{17} - 722*t^{16} + 671*t^{15} - 593*t^{14} + 470*t^{13} - 402*t^{12} + 325*t^{11} - 227*t^{10} + 169*t^{9} - 136*t^{8} + 83*t^{7} - 56*t^{6} + 41*t^{5} - 24*t^{4} + 12*t^{3} - 6*t^{2} + 4*t - 1 )$$
Taylor Expansion of the Growth Function

\[ PS(t) := R!gf; \]
\[ \text{PS}; \]

\[
1 + 4t + 12t^2 + 36t^3 + 108t^4 + 288t^5 + 792t^6 + 2164t^7 + 5840t^8 + 15892t^9 + 43164t^{10} + 117144t^{11} + 318132t^{12} + 863694t^{13} + 2344668t^{14} + 6365680t^{15} + 17282348t^{16} + 46919944t^{17} + 127384668t^{18} + 345839040t^{19} + 938923320t^{20} + 2549100468t^{21} + 6920597292t^{22} + 18788854468t^{23} + 51010209656t^{24} + 138488547448t^{25} + 375985071642t^{26} + 1020768699264t^{27} + 2771303459880t^{28} + 7523862091500t^{29} + 20426666923956t^{30} + 55456737328132t^{31} + 150560503742828t^{32} + 408759446927104t^{33} + 1109748448270104t^{34} + 3012876223082520t^{35} + 8179712394094716t^{36} + 22207249783071348t^{37} + O(t^{38})
\]
The Outcome

- There are 10,986 hyperbolic manifolds in the Hodgson-Weeks census.
- Automatic structure attempted for the first 5,800 manifold groups and succeeded in 5,390 cases (93% of the total).
- The largest number of equations produced by KB in a successful run is 1.1 million.
- The largest word acceptor among the successful cases has 45,516 states.
- The automatic structures are available as a Magma database.
What’s Next?

We want people to make use of these automatic structures. Attendees of this workshop are entitled to a *Conference Version* of Magma.

Participants located at a US university should get their version via the Simons-Magma scheme.

To arrange to set up the Conference Version please email magma@maths.usyd.edu.au.
FP-Groups with Solvable Word Problem

- Abelian groups.
- $p$-groups: power-conjugate presentation.
- Nilpotent groups: polycyclic presentation.
- Finite soluble groups: polycyclic presentation.
- Infinite polycyclic groups: (soluble groups)
- Automatic groups, eg Coxeter groups.
- Braid groups.
- Free groups.
Soluble Quotient Algorithms

Given an fp-group $G$ we would like to know if it belongs to one of the above classes and if so compute a solution to its word problem.

Algorithms for computing polycyclic quotients and their specialisations solve part of the problem.

- Abelian quotient algorithm: Smith normal form.
- Nilpotent quotient algorithm. (Chen-Fox-Lyndon; Sims, Nickel)
- Soluble quotient algorithm. (Wamsley, Howse, Johnson, Leedham-Green, Plesken, Niemeyer, Bruckner)
- Polycyclic quotient algorithm (Baumslag-Cannonito-Miller; Sims, Lo, Eick, ...).
Abelian Quotient Algorithm

We are interested in finding the largest abelian quotient $AQ$ of an fp-group $G$.

$AQ$ is found by abelianising the relations of $G$ and computing the Smith Normal Form of the resulting matrix.

For small examples this is very straightforward.

However, care is needed if our presentation has tens of thousands of generators and relations.
Smith Normal Form: Sparse Case

Main idea: Apply *Markowitz pivoting* to reduce the size of the matrix $A$:

- Successively choose a unit pivot $p = A_{i,j}$ such that $w = (w_r - 1) \cdot (w_c - 1)$ is minimal, where $w_r, w_c$ are the weights of row $r$ and column $c$ of $A$, respectively; clear all rows & columns by pivot $p$ and swap $p$ to next diagonal position.

- The weight $w$ counts the number of non-zero entries introduced at each reduction, so choosing the smallest possible $w$ means minimal fill-in.

- Stop when density reaches 50% and switch to dense algorithm.

- The most expensive part of the algorithm is maintaining the weights and the search for the next best pivot.

Similar to Havas/Holt/Rees algorithm (JSC 1993) but it is essential to use a sparse matrix data structure and switch to dense algorithm before the matrix gets completely dense.
Smith Normal Form: Dense Case

Given dense matrix $A$:

1. Find the rank $r$ of $A$ and compute several $r \times r$ subdeterminants and let $g$ be the GCD of these.

2. If $g$ is smooth, use Lübeck algorithm (JSC 2002) for each prime $p$ dividing $g$ to compute the multiplicities of $p$ in each non-trivial entry of SNF (uses fast nullspace computations mod $p$ for each $p$).

3. Otherwise (generic case): simply compute the Hermite Normal Form (HNF), transpose and repeat until the matrix is diagonal; this gives the SNF.

4. HNF: special modular algorithm in the spirit of Micciancio & Warinschi (ISSAC 2001): attempts to find $s$ close to rank $r$ so that the lattice generated by first $s$ columns has small determinant $D$; the HNF of these columns is computed very quickly (modulo $2 \cdot D$) and HNF of $A$ is then fully constructed by modular techniques.
Smith Normal Form: Example

Compute SNF of sparse matrix (arising from abelianisation of the kernel of an epimorphism with image $L_2(41)$):

1. Sparse input matrix: $68880 \times 34441$, density 0.018%, 430296 non-zero entries.

2. Sparse reduction: reduces to dense matrix $D$: $19011 \times 4140$, density 50.0% (15.3 secs)

3. SNF of dense $D$: uses GCDs of 10 subdeterminants with maximal rank 4057, then iterative modular Hermite NF (2475 secs).

4. Resulting abelian structure:
   \[
   \mathbb{Z}_2^2 \times \mathbb{Z}_4^{40} \times \mathbb{Z}_8^5 \times \mathbb{Z}_{328}^{36} \times \mathbb{Z}_{656}^5 \times \mathbb{Z}_{1312}^{38} \times \mathbb{Z}_{19246948714239968} \times \\
   \mathbb{Z}_{96234743571199840} \times \mathbb{Z}_{26502049500299981064210777263546080} \times \\
   \mathbb{Z}_{1086584029512299223632641867805389280} \times \mathbb{Z}_{83}.
   \]

5. Total time: 2490 secs.
The $p$-quotient Algorithm

Let $G$ be a finite $p$-group.

Every finite $p$-group $G$ has a presentation of the form:

$$\langle a_1, \ldots, a_n \mid a_i^p = u_{ii}, \ 1 \leq i \leq n, \ [a_k, a_j] = u_{jk}, \ 1 \leq j < k \leq n \rangle$$

where $u_{jk}$ is a word in $a_{k+1}, \ldots, a_n$ for $1 \leq j \leq k \leq n$.

This is called a power-commutator presentation (or pc-presentation) for $G$.

The generators are called pc-generators.
The $p$-quotient Algorithm

Each element of $G$ has a normal form as a word in the pc-generators.

The normal form is found using the defining relations as rewrite rules.

The $p$-quotient algorithm computes successively larger $p$-quotients of $G$ with respect to the terms of its $p$-central series.
Polycyclic Groups

A polycyclic group is a group $G$ with a subnormal series

$$G = G_1 > G_2 > .. > G_{n+1} = 1$$

in which every quotient $G_i/G_{i+1}$ is cyclic.

Every polycyclic group $G$ has a presentation of the form

$$\langle a_1, \ldots, a_n | a_i^{m_i} = w_{i,i} \quad (i \in I),
      a_j^{a_i} = w_{i,j} \quad (1 \leq i < j \leq n),
      a_j^{a_i^{-1}} = w_{-i,j}, \quad 1 \leq i < j \leq n, \quad i \notin I \rangle$$

(i) $I$ is a subset of $\{1, \ldots, n\}$.
(ii) $m_i > 1$ for $i \in I$, and
(iii) $w_{i,j} = a_i^{l(i, j, |i|+1)} \ldots a_n^{l(i, j, n)}$ with $0 \leq l(i, j, k) < m_k$ if $k \in I$. 
For $1 \leq i \leq n$, let $G_i$ be the subgroup of $G$ generated by $a_i, \ldots, a_n$ and define $G_{n+1}$ to be the trivial group.

(i) $|G_i/G_{i+1}| = m_i$ when $i \in I$

(ii) $G_i/G_{i+1}$ is infinite when $i \notin I$.

Thus, polycyclic groups are built up from the identity subgroup by a finite number of cyclic extensions.

They are solvable and all subgroups are finitely generated.

Every finitely generated nilpotent group is polycyclic.
Nilpotent Quotient Algorithm

\[ G\langle a,b \rangle := \text{Group}\langle a,b | a*b*a^{-1} = b^4 \rangle; \]

\[ \text{AbelianQuotient}(G); \]
Abelian Group isomorphic to \( \mathbb{Z}/3 + \mathbb{Z} \)

\[ N, f := \text{NilpotentQuotient}(G, 6); \]
\[ N; \]
GrpGPC : \( N \) of infinite order on 7 PC-generators
PC-Relations:
\[
\begin{align*}
N.2^3 &= N.3^2 \times N.4^2 \times N.5 \times N.7^2, \\
N.3^3 &= N.4^2 \times N.5^2 \times N.6, \\
N.4^3 &= N.5^2 \times N.6^2 \times N.7, \\
N.5^3 &= N.6^2 \times N.7^2, \\
N.6^3 &= N.7^2, \\
N.7^3 &= \text{Id}(N),
\end{align*}
\]
Nilpotent Quotient Algorithm (cont)

\[ N.2^{N.1} = N.2 \ast N.3, \]
\[ N.2^{(N.1^{-1})} = N.2 \ast N.3^2 \ast N.4^2 \ast N.5 \ast N.7^2, \]
\[ N.3^{N.1} = N.3 \ast N.4, \]
\[ N.3^{(N.1^{-1})} = N.3 \ast N.4^2 \ast N.5^2 \ast N.6, \]
\[ N.4^{N.1} = N.4 \ast N.5, \]
\[ N.4^{(N.1^{-1})} = N.4 \ast N.5^2 \ast N.6^2 \ast N.7, \]
\[ N.5^{N.1} = N.5 \ast N.6, \]
\[ N.5^{(N.1^{-1})} = N.5 \ast N.6^2 \ast N.7^2, \]
\[ N.6^{N.1} = N.6 \ast N.7, \]
\[ N.6^{(N.1^{-1})} = N.6 \ast N.7^2 \]

> [ Order(N.i) : i in [1..7] ];
[ Infinity, 729, 243, 81, 27, 9, 3 ]

> N, f := NilpotentQuotient(G, 12);
> [ Order(N.i) : i in [1..13] ];
[ Infinity, 531441, 177147, 59049, 19683, 6561, 2187, 729, 243, 81, 27, 9, 3 ]
Consider the group

\[ G := \langle b^a = bc, \quad b^{a^{-1}} = b \ast c^{-1} \rangle \]

\[
\begin{align*}
> &\quad F<a,b,c> := \text{FreeGroup}(3); \\
> &\quad \text{rels} := [ b^a = b \ast c, \quad b^{a^{-1}} = b \ast c^{-1} ]; \\
> &\quad G<a,b,c> := \text{quo}<\text{GrpGPC}: F \mid \text{rels}>; \\
> &\quad \text{IsAbelian}(G); \\
&\quad \text{false} \\
> &\quad \text{AbelianQuotient}(G); \\
&\quad \text{Abelian Group isomorphic to Z + Z} \\
&\quad \text{Defined on 2 generators (free)} \\
> &\quad \text{DerivedGroup}(G); \\
&\quad \text{GrpGPC of infinite order on 1 PC-generators} \\
&\quad \text{PC-Relations:}
\end{align*}
\]
Non-soluble Quotients

- Epimorphisms onto a finite group (Derek Holt)
- Simple quotients (Derek Holt)
- L2, L3 and U3 quotients (Plesken, Fabianska, Jambor)
Homomorphisms

B := BraidGroup(GrpFP, 4);
G := PSL(2,16);
P := HomomorphismsProcess(B, G : Surjective := false, TimeLimit := 10);
while not IsEmpty(P) do
  if DefinesHomomorphism(P) then
    f := Homomorphism(P);
    Im := Image(f);
    if IsMaximal(G,Im) then
      print "Found image which is maximal subgroup"
      break;
    end if;
  end if;
  end if;
  if IsValid(P) then
    NextElement(~P);
  else
    print "Limit has been reached"
    break;
  end if;
end while;
Homomorphisms

Found image which is maximal subgroup
>
> \( f; \)

Homomorphism of GrpFP: B into GrpPerm: G, induced by

\[
\begin{align*}
B.1 \mid & \rightarrow (1, 4, 3, 6, 12)(2, 11, 9, 16, 7) \\
& (5, 17, 8, 15, 13) \\
B.2 \mid & \rightarrow (1, 4, 2, 10, 16)(3, 11, 9, 12, 14) \\
& (5, 15, 8, 13, 17) \\
B.3 \mid & \rightarrow (1, 4, 3, 6, 12)(2, 11, 9, 16, 7) \\
& (5, 17, 8, 15, 13)
\end{align*}
\]

> \#Im;

60

> CompositionFactors(Im);

G
\mid Alternating(5)
1
\textbf{L2-Quotients}

\begin{verbatim}
> G := Group< a,b | a^2, b^3, (a*b)^16, (a,b)^11 >;
> L2Quotients(G);
[  PGL(2,23),
  PGL(2,23),
  PGL(2,463)
]
\end{verbatim}
L2-Quotients

> G := Group< a,b,c | a^3, b^7, (a*b)^2, (a*c)^2, (b*c) 
> L2Quotients(G);

[ L_2(infty^6) ]

> H := Group<a,b,c | a^3, (a,c)=(c,a^-1), a*b*a=b*a*b, a*b*a*c^-1=c*a*b*a >;
> L2Quotients(H);

[ L_2(3^infty) ]

> K := Group< a,b | a^3*b^3 >;
> L2Quotients(K);

[ L_2(infty^infty) ]
L3-Quotients

\[
\text{G := Group< a, b | a^2, b^3, (a*b)^18, (a,b)^16 >;}
\]

\[
\text{L3Quotients(G);}\\
\text{[}
\text{PGU}(3, 71), \\
\text{U}_3(1889), \\
\text{PGU}(3, 17), \\
\text{PGU}(3, 17), \\
\text{PGU}(3, 17), \\
\text{PGL}(3, 19)
\text{]}\]
Let $\phi$ be an epimorphism from fp-group $G$ to a finite group $H$. Subgroups of $G$ of small (finite) index can be found as follows:-

1. Low index subgroups algorithm.
2. Low index normal subgroups algorithm.
3. Preimages wrt $\phi$ of subgroups of $H$.
4. Kernel of $\phi$.
5. Given subgroups represented by their coset tables, further subgroups can be found by computing intersections, normalisers, conjugates, normal closures and cores.
Low Index Subgroups

> G := Group< x, y | x^2, y^3, (x*y)^7 >;
> time #LowIndexSubgroups(G, 40);
Time: 0.020
257
> time #LowIndexSubgroups(G, 60);
Time: 0.410
5122
> time #LowIndexSubgroups(G, 80);
Time: 12.360
115940
> time #LowIndexSubgroups(G, 100);
Time: 380.590
3037698

> time n := LowIndexNormalSubgroups(G, 100);
Time: 63.830
> #n;
1
Practical Criteria for Establishing Non-Finiteness

▶ Deficiency of a presentation is negative.
▶ Abelian quotient of a subgroup or quotient group of $G$ is infinite. (Can use either SNF or Holt-Plesken condition).
▶ A polycyclic quotient algorithm detects an infinite section in $G$ or some subgroup of $G$.
▶ Golod-Safarevic finiteness criteria for $p$-groups as recast by Mike Newman violated.
▶ The kernel of some epimorphism found by a soluble or non-soluble quotient algorithm has an infinite abelian quotient.
▶ The automatic structure of $G$ enables non-finiteness to be decided.

**Theorem:** (Golod-Safarevic) Let $G$ be a finite $p$-group with $b$ generators and $r$ relations. Then $r > b^2/4$. 
Example: Consider the family

\[ G = \langle a, b | a^3, b^7, (a \ast b)^2, (a, b)^r \rangle \]

*** The case \( r = 18 \) ***
Subgroup \( H \) has index 9 in \( G \)
Its core \( K \) has index 56 in \( H \)
\( K \) has an infinite abelian quotient
Time: 0.070

*** The case \( r = 20 \) ***
Subgroup \( H \) has index 7 in \( G \)
Its core \( K \) has index 24 in \( H \)
\( K \) has infinite 5-quotient by Golod-Safarevic
Time: 0.270

*** The case \( r = 21 \) ***
Subgroup \( H \) has index 14 in \( G \)
Its core \( K \) has index 78 in \( H \)
\( K \) has infinite 3-quotient by Golod-Safarevic.
Time: 0.070
*** The case $r = 24$ ***
Subgroup $H$ has index 7 in $G$
The core $K$ of $H$ has index 24 in $H$
$K$ has infinite 3-quotient by Golod-Safarevic.
Time: 0.280

*** The case $r = 25$ ***
Subgroup $H$ has index 56 in $G$
Coset action of $G$ on $H$ gives $P = \text{Alt}(56)$
$K$ is preimage of subgroup of index 1512 of $P$
$K$ has an infinite abelian quotient
Time: 14.590

*** The case $r = 33$ ***
Subgroup $H$ has index 57 in $G$
Class 1 3-quotient $Q$ of $H$ has order 9
$K$ is the kernel of $Q$
$K$ has an infinite abelian quotient
Time: 2.340
*** The case $r = 38$ ***
Subgroup $H$ has index 21 in $G$
Coset action of $G$ on $H$ gives $P = \text{Alt}(21)$
$K$ is preimage of subgroup of index 210 of $P$
$K$ has infinite 2-quotient by Golod-Safarevic
Time: 0.410

*** The case $r = 42$ ***
Subgroup $H$ has index 14 in $G$
The core $K$ of $H$ has index 78 in $H$
$K$ has infinite 7-quotient by Golod-Safarevic
Time: 0.410

*** The case $r = 46$ ***
Subgroup $H$ has index 94 in $G$
Class 1 7-quotient $Q$ of $H$ has order 49
$K$ is the kernel of $Q$
$K$ has an infinite abelian quotient
Time: 790.640