Well-Posedness of Space-Discrete Diffusion Filters with Negative Diffusivities

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Introduction

- In the last 20 years, many partial differential equations (PDEs) and variational approaches have been proposed for enhancing digital images.
- The continuous formulation of these models offer advantages, e.g.
  - transparent and compact models
  - invariances (under rotation, affine transformations, ...)
- However, some of the most interesting models are difficult to analyse in the continuous setting due to well-posedness problems:
  - Perona–Malik filter (1990)
  - forward-and-backward (FAB) diffusion (Gilboa/Sochen/Zeevi 2002)
  - stabilised inverse linear diffusion (Osher/Rudin 1990)
  - shock filtering (Osher/Rudin 1990)
- Often these filters work well in practise, but lack a sound continuous theory.
- What about well-posedness in the discrete setting ?

Well-Posedness Results for Space Discretisations

Perona–Malik Filter (1990):

$$\partial_t u \;=\; \mathrm{div}\,\left(g(|\boldsymbol{\nabla} u|^2)\,\boldsymbol{\nabla} u\right)$$

with a decreasing, nonnegative diffusivity function g.
Rapid decay may lead to nonmonotone fluxes and contrast enhancement.

- **W. 1998:**
  space-discrete and fully discrete theory for smooth positive diffusivities
- **W. / Benhamouda 1998:**
  monotonicity preservation of 1-D explicit scheme (allows staircasing)
- **Pollak / Willsky / Krim 2000:**
  space-discrete well-posedness for singular nonnegative diffusivities (SIDEs)

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Stabilised Inverse Linear Diffusion (Osher / Rudin 1990)

$$\partial_t u \;=\; \begin{cases} -\Delta u & \text{for } |\boldsymbol{\nabla} u| \neq 0, \\ 0 & \text{for } |\boldsymbol{\nabla} u| = 0. \end{cases}$$

- **Osher / Rudin 1990:**
  no continuous well-posedness theory, but stable minmod discretisation

Shock Filtering (Osher / Rudin 1990)

$$\partial_t u \;=\; -\mathrm{sgn}(\Delta u)\,|\boldsymbol{\nabla} u|$$

- **Welk / W. / Galić 2007:**
  for 1-D space discretisation:
  well-posedness, extremum principle, analytic solution

FAB-Diffusion (Gilboa / Sochen / Zeevi 2002)

- essentially Perona-Malik diffusion

$$\partial_t u \; = \; \text{div} \left(g(|\boldsymbol{\nabla} u|^2)\, \boldsymbol{\nabla} u \right)$$

  but with a C^1 diffusivity that may attain positive and *negative* values, e.g. Bogdan's diffusivity

$$g(|\boldsymbol{\nabla} u|^2) \; = \; 2 \exp\left(\frac{|\boldsymbol{\nabla} u|^2}{\lambda_1^2}\right) \; - \; \exp\left(\frac{|\boldsymbol{\nabla} u|^2}{\lambda_2^2}\right) \qquad (\lambda_2 > \lambda_1 > 0).$$

- can outperform Perona-Malik diffusion
- no well-posedness theory so far, neither continuous nor discrete
- pessimism w.r.t. extremum principle (standard implementations violate it)

Is there a (space-)discrete well-posedness theory ?

Outline

- Space-Discrete Diffusion Theory
- Application to Space-Discrete FAB Diffusion
- Fully Discrete Results
- Summary and Outlook

Outline

Space-Discrete Diffusion Theory

(W. 1998)

standard spatial discretisation of diffusion equation

$$\begin{aligned} \partial_t u &= \operatorname{div}\left(g(|\boldsymbol{\nabla} u|^2)\, \boldsymbol{\nabla} u\right) \\ &= \partial_x \left(g(|\boldsymbol{\nabla} u|^2)\, \partial_x u\right) \;+\; \partial_y \left(g(|\boldsymbol{\nabla} u|^2)\, \partial_y u\right) \end{aligned}$$

in some inner pixel (i,j) yields the ordinary differential equation

$$\begin{aligned} \frac{du_{i,j}}{dt} &= \frac{1}{h_1}\left(\frac{g_{i+1,j}+g_{i,j}}{2}\,\frac{u_{i+1,j}-u_{i,j}}{h_1} - \frac{g_{i,j}+g_{i-1,j}}{2}\,\frac{u_{i,j}-u_{i-1,j}}{h_1}\right) \\ &+ \frac{1}{h_2}\left(\frac{g_{i,j+1}+g_{i,j}}{2}\,\frac{u_{i,j+1}-u_{i,j}}{h_2} - \frac{g_{i,j}+g_{i,j-1}}{2}\,\frac{u_{i,j}-u_{i,j-1}}{h_2}\right). \end{aligned}$$

Space-Discrete Diffusion Theory (2)

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More Compact Notation

Representing pixel (i,j) by a single index $k(i,j)$ leads to

$$\frac{du_k}{dt} = \sum_{n=1}^{2} \sum_{l \in \mathcal{N}_n(k)} \frac{g_l + g_k}{2h_n^2} (u_l - u_k),$$

where $\mathcal{N}_n(k)$ are the neighbours of pixel k in n-direction
(boundary pixels may have less neighbours).

This can be written as a system of ordinary differential equations (ODEs):

$$\frac{d\boldsymbol{u}}{dt} = A(\boldsymbol{u})\,\boldsymbol{u},$$

where $\boldsymbol{u} = (u_1, ..., u_N)^\top$, and the $N \times N$ matrix $A(\boldsymbol{u}) = (a_{k,l}(\boldsymbol{u}))$ satisfies

$$a_{k,l} := \begin{cases} \frac{g_k+g_l}{2h_n^2} & (l \in \mathcal{N}_n(k)), \\ -\sum_{n=1}^{2} \sum_{l \in \mathcal{N}_n(k)} \frac{g_k+g_l}{2h_n^2} & (l = k), \\ 0 & (\text{else}). \end{cases}$$

Space-Discrete Diffusion Theory (3)

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Assumptions

Let $\boldsymbol{f} \in \mathbb{R}^N$ and consider the ODE system

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt} &= A(\boldsymbol{u})\,\boldsymbol{u}, \\ \boldsymbol{u}(0) &= \boldsymbol{f} \end{aligned}$$

where $A = (a_{i,j})$ is assumed to have the following properties:

- **(S1) Smoothness:**
  A is a Lipschitz-continuous function of $\boldsymbol{u}$, for every bounded subset of $\mathbb{R}^N$.
- **(S2) Symmetry:** $a_{i,j}(\boldsymbol{u}) = a_{j,i}(\boldsymbol{u})$
- **(S3) Vanishing Row Sums:** $\sum_j a_{i,j}(\boldsymbol{u}) = 0 \quad \forall\, i$
- **(S4) Nonnegative Off-Diagonals:** $a_{i,j}(\boldsymbol{u}) \geq 0 \quad \forall\, i \neq j$
- **(S5) Irreducibility:**
  For any i, j there exist k_0,...,k_r with $k_0 = i$ and $k_r = j$ such that $a_{k_p,k_{p+1}} \neq 0$ for $p = 0$,...,$r-1$. ("There is some way from pixel i to pixel j")

Space-Discrete Diffusion Theory (4)

Results

(a) **Well-Posedness** (requires (S1), (S3), (S4))
Existence of a unique solution which depends continuously on $\boldsymbol{f}$ and the right hand side of the ODE system

(b) **Average Grey Level Invariance** (requires (S2), (S3))

(c) **Maximum–Minimum Principle** (requires (S3), (S4))

$$\min_j f_j \;\leq\; u_i(t) \;\leq\; \max_j f_j \qquad \forall i,\; \forall t > 0.$$

(d) **Lyapunov Functions** (requires (S2)–(S4))

$$V(t) \;:=\; \sum_i r(u_i(t))$$

is a Lyapunov function for all convex $r \in \mathrm{C}^1$: decreasing and bounded from below.
Implies decreasing p-norms and central moments, and increasing entropy.

(e) **Convergence to a Constant Steady-State** (requires (S2)–(S5))

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Application to Space-Discrete FAB Diffusion

The Good News First

- Space-discrete FAB diffusion satisfies
  - (S1) smoothness
  - (S2) symmetry
  - (S3) vanishing row sums
- This allows to establish
  - local well-posedness (using (S1))
  - average grey level invariance (using (S2), (S3))

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Now the Bad News

- Space-discrete FAB diffusion does not satisfy
  - (S4) nonnegative off-diagonals:
    negative diffusivitives give negative off-diagonals
  - (S5) irreducibility:
    diffusivities and off-diagonals may become 0
- This affects
  - extremum principle (requires (S4))
  - global well-posedness (requires (S4))
  - Lyapunov functionals (requires (S4))
  - convergence to a constant steady state (requires (S4), (S5))

Is there hope for an extremum principle and global well-posedness ?

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Relaxation of (S4) to Extrema

Replacing

- (S4) nonnegative off-diagonals:
  $a_{i,j}(\boldsymbol{u}) \geq 0$ for all i, j with $i \neq j$

by the less restrictive condition

- (S4a) nonnegative off-diagonals at extrema:
  $a_{i,j}(\boldsymbol{u}) \geq 0$ for all j with $j \neq i$ if u has an extremum in i.

still allows to establish an extremum principle:

$$\min_j f_j \;\leq\; u_i(t) \;\leq\; \max_j f_j \qquad \forall\, i,\; \forall\, t > 0.$$

Does Space-Discrete FAB Diffusion Satisfy (S4a) ?

- (S4a) requires nonnegative diffusivity $g_{i,j}$ if $u_{i,j}$ is an extremum.
- possible if an extremum gives a vanishing discrete gradient
- not satisfied for standard discretisations of $g(|\boldsymbol{\nabla} u|^2)$ such as

$$g_{i,j} \;:=\; g\left(\left(\frac{u_{i+1,j}-u_{i-1,j}}{2h_1}\right)^2 + \left(\frac{u_{i,j+1}-u_{i,j-1}}{2h_2}\right)^2\right)$$

- satisfied for the *nonstandard discretisation*

$$g_{i,j} \;:=\; g\left(\max\left(\frac{u_{i+1,j}-u_{i,j}}{h_1}\cdot\frac{u_{i,j}-u_{i-1,j}}{h_1},\, 0\right) + \max\left(\frac{u_{i,j+1}-u_{i,j}}{h_2}\cdot\frac{u_{i,j}-u_{i,j-1}}{h_2},\, 0\right)\right).$$

Theorem 1 (Space-Discrete FAB Diffusion)

The space discretisation of FAB diffusion with $g(0) > 0$ and $g \in C^1[0, \infty)$ is well-posed, satisfies a maximum–minimum principle and average grey level invariance, if the diffusivity is evaluated with the nonstandard finite difference approximation

$$g_{i,j} \;:=\; g\left(\max\left(\frac{u_{i+1,j}-u_{i,j}}{h_1}\cdot\frac{u_{i,j}-u_{i-1,j}}{h_1},\,0\right) + \max\left(\frac{u_{i,j+1}-u_{i,j}}{h_2}\cdot\frac{u_{i,j}-u_{i,j-1}}{h_2},\,0\right)\right).$$

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Fully Discrete Results

Discretisation

Let us consider a 1-D explicit scheme for $u_t = \partial_x(g(u_x^2)\, u_x)$:

$$\frac{u_i^{k+1} - u_i^k}{\tau} = \frac{1}{h}\left(\frac{g_{i+1}^k + g_i^k}{2}\,\frac{u_{i+1}^k - u_i^k}{h} - \frac{g_i^k + g_{i-1}^k}{2}\,\frac{u_i^k - u_{i-1}^k}{h}\right)$$

where the diffusivity uses the nonstandard discretisation

$$g_i^k = g\left(\max\left(\frac{u_i^k - u_{i-1}^k}{h} \cdot \frac{u_{i+1}^k - u_i^k}{h}, 0\right)\right)$$

The input signal $\boldsymbol{f}$ serves as initialisation:

$$u_i^0 = f_i \qquad (i = 1, ..., N).$$

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Assumptions

- The grey values of $\boldsymbol{f} = (f_i)$ are restricted to a finite interval of length R.
- There exist $c_1 > c_2 > 0$ with $g(0) = c_1$ and $g(s^2) > -c_2$ for all s^2.
- A positive ω exists such that $g(s^2) > c_2$ holds for all s, $0 < s < \omega R$.
- The time step size satisfies $\tau < \frac{\omega^2 h^4}{c_1 + c_2 + 2c_1\omega^2 h^2}$.

Consequences

- **Maximum-Minimum Principle**

$$\min_j f_j \le u_i^k \le \max_j f_j \qquad \forall\, i,\ \forall\, k \in \mathbb{N}_0.$$

- **Total Variation Reduction**

$$\sum_{i=1}^{N-1} |u_{i+1}^{k+1} - u_i^{k+1}| \le \sum_{i=1}^{N-1} |u_{i+1}^k - u_i^k| \qquad \forall\, k \in \mathbb{N}_0.$$

Fully Discrete Results (4)

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Main Steps in the Proof:

1. A local maximal pixel does not increase.
2. A neighbour pixel of a local maximum cannot increase in excess of this maximum.
3. No new extrema are generated around existing extrema.
4. Monotonicity is preserved in segments without extrema.

Fully Discrete Results (4)

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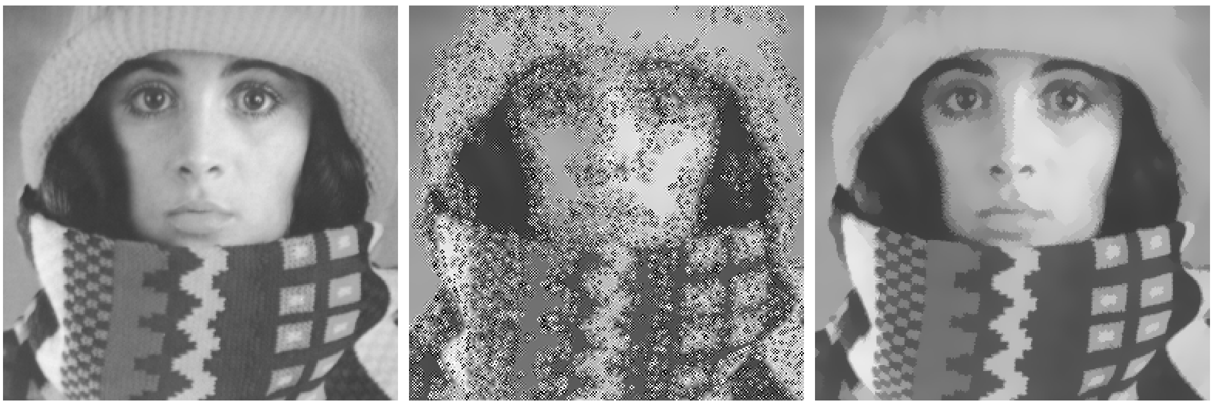

(a) Left: Original image, 256×256 pixels. **(b) Middle:** After filtering with an explicit scheme for FAB diffusion with Bogdan diffusivity ($\lambda_1 = 4$, $\lambda_2 = 10$) and standard discretisation (100 iterations with $\tau = 0.1$). **(c) Right:** Same with nonstandard discretisation.

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Summary

It pays off to analyse space-discrete formulations of PDEs:

- showed that FAB diffusion becomes well-posed under space discretisation and satisfies an extremum principle
- requires nonstandard discretisation in the diffusivity
- results carry over to fully discrete explicit scheme in 1D

Outlook

- fully discrete properties in higher dimensions ?
- (semi-)implicit schemes ?

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References

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Download: `www.mia.uni-saarland.de/weickert/publications.shtml`

Thank you for your attention!