Homology of moduli spaces of linkages in high-dimensional euclidean space

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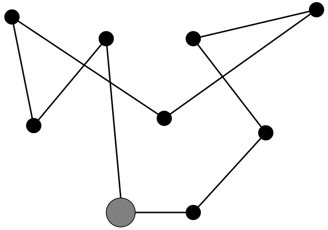
Applied and Computational Topology: ATMCS 5
Edinburgh ICMS, 04.07.2012

Introduction

Given $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{R}^n$ with $\ell_i > 0$, we want to look at closed linkages in some euclidean space $\mathbb{R}^d$ up to rotation

$$\mathcal{M}_d(\ell) \quad = \quad \left\{ (z_1, \dots, z_n) \in (S^{d-1})^n \,\middle|\, \sum_{i=1}^n \ell_i z_i = 0 \right\} \Big/ SO(d)$$

and study how the topology depends on the length vector ℓ.



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$$H_J = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{j \in J} x_j = \sum_{i \notin J} x_i \right\}$$

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Also, if we define for $\sigma \in \Sigma_n$ the length vector ℓ^σ by

$$\ell^\sigma = (\ell_{\sigma 1}, \ldots, \ell_{\sigma n})$$

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One may therefore ask whether these are the only conditions leading to the same spaces.

Theorem (Farber, Hausmann, S)

Let ℓ, ℓ' be generic, such that $\mathcal{M}_d(\ell) \simeq \mathcal{M}_d(\ell')$ for $d = 2$ or 3.

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2. *If $d = 3$, and $n \geq 5$, then ℓ and ℓ' are in the same chamber up to permutation.*

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For $d \geq 4$ we get

$$\mathcal{M}_d(\ell) \;\cong\; D^2 \;\cong\; \mathcal{M}_d(\ell').$$

The proof of the Theorem is based on knowledge of $H^*(\mathcal{M}_d(\ell);\mathbb{Z})$ for $d = 2, 3$, or rather identifying an appropriate sub ring ($d = 2$), respectively, quotient ring ($d = 3$), for which an isomorphism problem can be solved (Gubeladze, '98).

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Kendall, Barden, Carne and Le have calculated the homology for $\ell = (1, \ldots, 1, n-2) \in \mathbb{R}^n$. They also show that in general these spaces do not satisfy Poincaré duality.

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and also let

$$a_k(\ell) \quad = \quad |\mathcal{S}_k(\ell)|.$$

The Betti numbers of $\mathcal{M}_d(\ell)$ can be expressed in terms of these for $d = 2, 3$.

The space of chains

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Define $f : \mathcal{C}_d(\ell) \to \mathbb{R}$ by

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where $p_1 : S^{d-1} \to \mathbb{R}$ is projection to the first coordinate. This is $SO(d-1)$ equivariant, and Morse-Bott.

The minimum is obtained at $\mathcal{C}_d(\ell^-) \subset \mathcal{C}_d(\ell)$, and the maximum is obtained at $\mathcal{C}_d(\ell^+) \subset \mathcal{C}_d(\ell)$, where

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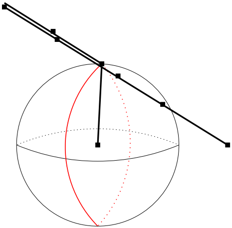

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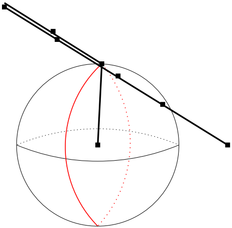

The index of such a critical sphere is $(n - 3 - |J|)(d - 1)$.

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The Morse-Bott function determines a filtration

$$\emptyset = \mathcal{M}^0 \subset \mathcal{M}^1 \subset \cdots \subset \mathcal{M}^m = \mathcal{M}_d(\ell)$$

such that $\mathcal{M}^{i+1}/\mathcal{M}^i \simeq X_d^k/\partial X_d^k$, where X_d^k is a quotient of a disc, and k corresponds to the index of a critical point.

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For example, for $d = 5$ and $k \geq 3$ one gets the following Poincaré polynomial:

$$P_5^k(t) \;=\; t^{4k-4\lfloor\frac{k-3}{2}\rfloor-3} \sum_{i=0}^{\lfloor\frac{k-3}{2}\rfloor} t^{4i}.$$

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for $m \geq 0$, we can express the Poincaré polynomial using Hausmann-Knutson as

$$P_3^\ell(t) \;=\; 1 + t^2 \cdot \sum_{i=0}^{k-2} (a_i - a_{n-2-i})\,(R_{n-4-i}(t^2) - R_{i-2}(t^2))$$

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where $k = \lfloor \frac{n+1}{2} \rfloor$ and $R_m(t) = 0$ for $m < 0$.

For $m \geq 0$ let

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Theorem

Let $\ell \in \mathbb{R}^n$ be a generic length vector with $a_0(\ell) \neq 0$. Then the Poincaré polynomial of $\mathcal{M}_5(\ell)$ is

$$P_5^\ell(t) = 1 + t^9 \cdot \sum_{i=0}^{k-2} (a_i - a_{n-2-i})\,(Q_{n-6-i}(t^4) - Q_{i-4}(t^4)),$$

where $k = \lfloor \frac{n+1}{2} \rfloor$ and $Q_m(t) = 0$ for $m < 0$.

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For even $d > 4$ we have the same kind of extension problem as in the case $d = 4$, but we can give a formula for the Euler characteristic, and also some Morse-type inequalities.