# Existence and uniqueness of models in persistent homology

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# Outline

Motivation and background

Existence of morphisms

Uniqueness for objects

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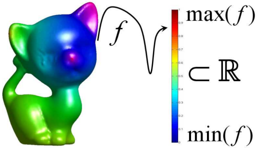
# Motivation

- In computer vision shape descriptors are used to analyze and compare objects starting from a geometric representation of an object
- Persistence modules are invariants that can be used as shape descriptors
    - they capture qualitative geometric information
    - they provide a parameter-free summary of the shape of an object via a multiscale approach
- Functoriality is central to understand relationships between different geometric objects

# What is a shape?

We consider for any $n \in \mathbb{N}$ the category $\mathscr{C}(n)$ so defined:

- Objects of $\mathscr{C}(n)$ are pairs $(X, f)$, where $X$ is a topological space and $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is a continuous function.




- A morphism between two objects $(X, f), (X', f')$ in $\mathscr{C}(n)$, is a continuous function $\gamma : X \to X'$ s. t. $f(x) \geq f'(\gamma(x))$ for all $x \in X$:

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & X' \\ & {}_{f}\searrow \;\; \geq \;\; \swarrow_{f'} & \\ & \mathbb{R}^n & \end{array}$$

(cf.,e.g.,M.Lesnick '12)

# Filtration of $X$ induced by $f$

Given $(X,f)$ in $\mathscr{C}(n)$, and $u \in \mathbb{R}^n$, consider $X_u = \bigcap_{i=1}^{n} f_i^{-1}((-\infty, u_i])$

If $u \leq v \in \mathbb{R}^n$, there is the inclusion $i_{u,v} \colon X_u \hookrightarrow X_v$

If $\gamma \colon (X,f) \to (X',f')$ is a morphism in $\mathscr{C}(n)$ we can consider the restriction of $\gamma$, $\gamma_u : X_u \to X'_u$.

Thus, starting from $\mathscr{C}(n)$, we can consider the category **n-Filt**:

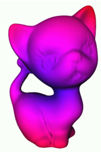
- Objects are families of nested spaces $(X_u)_{u \in \mathbb{R}^n}$ with inclusions $i_{u,v} \colon X_u \hookrightarrow X_v$ whenever $u \leq v \in \mathbb{R}^n$.
- Morphisms are families $(\gamma_u)_{u \in \mathbb{R}^n}$ of maps $\gamma_u : X_u \to X'_u$ such that if $u \leq v \in \mathbb{R}^n$ $i'_{u,v} \circ \gamma_u = \gamma_v \circ i_{u,v}$, that is

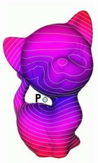
$$\begin{array}{ccc} X_u & \xrightarrow{\gamma_u} & X'_u \\ {\scriptstyle i_{u,v}} \big\downarrow & & \big\downarrow {\scriptstyle i'_{u,v}} \\ X_v & \xrightarrow{\gamma_v} & X'_v \end{array} .$$

## The persistent homology functor

So we obtain a functor $F_n : \mathscr{C}(n) \to$ **n-Filt**:

- $F_n(X, f) = \left( (X_u)_{u \in \mathbb{R}^n} \right)$
- $F_n(\gamma) = (\gamma_u)_{u \in \mathbb{R}^n}$






Now we can apply the homology functor to **n-Filt**.

### Definition

The *i-th persistent homology functor* is the composite functor $H_i \circ F$ where $H_i$ is the ordinary $i$th homology functor.

# Persistence modules

A *persistence module* **M** is a family $\{M_u\}_{u\in\mathbb{R}^n}$ of modules together with a family of homomorphisms $\{\iota_M(u,v) : M_u \to M_v\}_{u\le v\in\mathbb{R}^n}$ such that, for all $u \le v \le w \in \mathbb{R}^n$

$$\iota_M(u,w) = \iota_M(v,w) \circ \iota_M(u,v), \qquad \iota_M(u,u) = \mathrm{id}_{M_u}$$

Given two persistence modules **M** and **N**, a morphism from **M** to **N** consists of a collection of homomorphisms $\mathbf{h} = (h_u : M_u \to N_u)_{u\in\mathbb{R}^n}$ such that, for all $u \le v \in \mathbb{R}^n$, $\iota_N(u,v) \circ h_u = h_v \circ \iota_M(u,v)$:

$$\begin{array}{ccc} M_u & \xrightarrow{h_u} & N_u \\ {\scriptstyle i_{r,s}}\downarrow & & \downarrow{\scriptstyle i'_{r,s}} \\ M_v & \xrightarrow{h_v} & N_v \end{array}$$

**h** is an isomorphism whenever each $h_u$ is an iso.

# Persistence modules

The category of persistence modules with their morphisms will be denoted by $\mathscr{M}$.

By applying the persistent homology functor to objects and morphisms of $\mathscr{C}(n)$ we obtain objects and morphisms of $\mathscr{M}$.

Objects of $\mathscr{M}$ can always be constructed in this way [Carlsson&Zomorodian'09].

# Problems

- To what extent do morphisms between persistence modules reflect relationships between different geometric objects?
- To what extent do persistent homology groups allow for reconstruction?

Rephrased differently:

- Given a morphism between persistence modules, does a map between geometric objects always exist, inducing that morphism?
- Given a persistence module, does a geometric object uniquely exist, inducing that persistence module?

# Contributions

Existence problem:

- We show that algebraic morphisms between persistence modules may be not induced by maps between geometric objects
- We introduce the notion of geometric morphisms between persistence modules.
- We provide invariants, called $H_0$-trees, for geometric isomorphism classes.

Uniqueness problem:

- We restrict to the case of curves.
- $H_0$-trees allow to reconstruct curves up to isomorhpisms
- Bidimensional persistent Betti numbers of 0th homology also allow to reconstruct curves up to isomorhpisms

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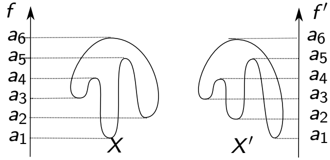
## A preliminary example

Two objects $(X,f),(X',f')$ in $\mathscr{C}(1)$ whose 0-th persistence modules in $\mathscr{M}$ are algebraically isomorphic:




$$
\begin{array}{ccc}
M_5 = < z_1, z_2, z_3 \mid z_1 = z_2 = z_3 > & \xrightarrow{h_5} & N_5 = < z_1', z_2', z_3' \mid z_1' = z_2' = z_3' > \\
\uparrow & & \uparrow \\
M_4 = < z_1, z_2, z_3 \mid z_1 = z_3 > & \xrightarrow{h_4} & N_4 = < z_1', z_2', z_3' \mid z_2' = z_3' > \\
\uparrow & & \uparrow \\
M_3 = < z_1, z_2, z_3 > & \xrightarrow{h_3} & N_3 = < z_1', z_2', z_3' > \\
\uparrow & & \uparrow \\
M_2 = < z_1, z_2 > & \xrightarrow{h_2} & N_2 = < z_1', z_2' > \\
\uparrow & & \uparrow \\
M_1 = < z_1 > & \xrightarrow{h_1} & N_2 = < z_1' >
\end{array}
$$

but no morphism between $(X,f)$ and $(X',f')$ is taken by the 0th persistence homology functor into this isomorphism.

# Geometric homomorphisms

Let $(X, f)$ and $(X', f')$ be two objects in $\mathscr{C}(n)$.

## Definition

A homomorphism $\mathbf{h}$ between the persistence modules $H_i \circ F(X, f)$ and $H_i \circ F(X', f')$ is called a *geometric homomorphism* if it belongs to the image of the persistent homology functor.

Questions:

- How to characterize geometric homomorphisms?
- Do persistence modules with geometric homomorphisms form a category?

# Algebraic invariants

Given a persistence module **M** consisting of vector spaces $(M_u)_{u\in\mathbb{R}^n}$ and homomorphisms $(\imath_M(u,v) : M_u \to M_v)_{u\leq v\in\mathbb{R}^n}$, its rank invariant is an integer-valued function $\rho_M$ of two variables $u \leq v \in \mathbb{R}^n$, defined by

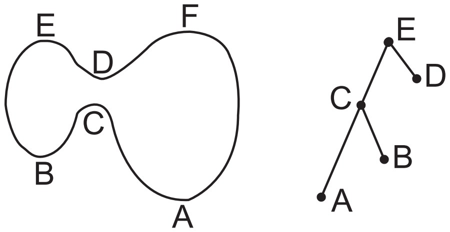
$$\rho_M(u,v) = rk(\imath_M(u,v))$$

- For $n = 1$, the rank invariant is a complete invariant for algebraic isomorphism of persistence modules with a finite presentation [Zomorodian&Carlsson'05]:
- For $n > 1$, the rank invariant is not a complete invariant for isomorphisms classes of persistence modules [Carlsson&Zomorodian'09].

# Geometric invariants

We now introduce an invariant, called $H_0$-tree, for geometric isomorphism of persistence modules in the case $n = 1$, that is for scalar functions $f : X \to \mathbb{R}$.

$H_0$-trees intuitively represent the fusion between pairs of connected components of the sub-level sets of the function $f$ along the filtration.



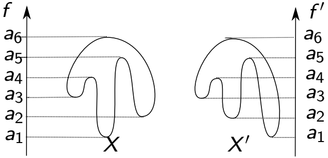
# $H_0$-tree

For a connected, compact manifold $X$ without boundary and a simple Morse function $f : X \to \mathbb{R}$, the $H_0$-tree of $f$ is a rooted binary tree labeled on the nodes defined as follows:

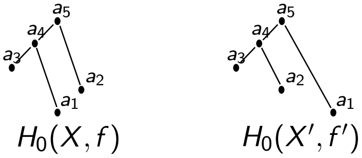
- the set of nodes is equal to the set of critical points of $f$ of index 0 or 1 if they merge two connected components.
- the label of a node $p$ is equal to $f(p)$;
- $p$ is a child of $q$ if $q$ has the lowest label among the nodes for which $f(p) < f(q)$ and $\iota(f(p), f(q)) : H_0(X_{f(p)}) \to H_0(X_{f(q)})$ takes the class of $p$ to that of $q$.

# $H_0$-tree in the preliminary example

Two objects non-isomorphic in $\mathscr{C}(1)$




whose $H_0$-trees are also non-isomorphic as labeled trees



# Geometric invariants

**Theorem**

*Let $X, X'$ be closed connected manifolds and $f : X \to \mathbb{R}, f' : X' \to \mathbb{R}$ be simple Morse functions.*
*If $h$ is a geometric isomorphism between $M = H_0 \circ F(X, f)$ and $N = H_0 \circ F(X', f')$, then the $H_0$-trees of $f$ and $f'$ are isomorphic as labeled trees.*

$H_0$-trees are not invariant for algebraic isomorphism classes of 0-persistence modules:

In the first example we have non-isomorphic $H_0$-trees whose persistence modules are algebraically isomorphic.

# What does uniqueness mean?

In general, it is false that if $(X, f)$ and $(X, g)$ have the same invariants then necessarily $f = g$.

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In general, it is false that if $(X, f)$ and $(X, g)$ have the same invariants then necessarily $f = g$.

But we may ask:

For given invariants, is $(X, f)$ realizing those invariants unique up to the relation $(X, f) \sim (Y, g)$ if and only if $f = g \circ h$ for some homeomorphism $h : X \to Y$?

# $H_0$-trees and uniqueness

$H_0$-trees are not complete invariants for geometric isomorphisms classes as can be seen by taking the first example using $-f$ and $-f'$ instead of $f$ and $f'$.

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But:

**Theorem**

*For $X = S^1$, simple Morse functions can be completely reconstructed up to $f$-preserving homeomorphisms, from the $H_0$-trees of $f$ and $-f$.*

# The rank invariant and uniqueness

**Theorem**

*Let $f, g : S^1 \to \mathbb{R}^2$ be generic functions (immersions, clean finitely many double points).*

*Let $\Sigma_2$ be the set of functions obtainable through finite composition of the reflections with respect to the coordinate axes.*

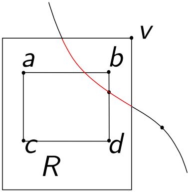
*If $\rho_{s \circ f}(u, v) = \rho_{s \circ g}(u, v)$ for every $u \prec v \in \mathbb{R}^2$ and every $s \in \Sigma_2$, then there exists a $C^1$-diffeomorphism $h : S^1 \to S^1$ such that $g \circ h = f$. Moreover, it is unique.*

# The rank invariant and uniqueness

*Sketch of proof:*

- Prove that $f(S^1) = g(S^1)$ by contradiction:

  If $u \in f(S^1) \setminus g(S^1)$, take $s \in \Sigma^2$ and $R$ s.t.
  $R \cap s \circ f(S^1)$ near $u$ looks like this:
  $R \cap s \circ g(S^1) = \emptyset$.




$$\text{rk } H_0^{b,v}(s \circ f) - \text{rk } H_0^{d,v}(s \circ f) - \text{rk } H_0^{a,v}(s \circ f) + \text{rk } H_0^{c,v}(s \circ f) = 1.$$

$$\text{rk } H_0^{b,v}(s \circ g) - \text{rk } H_0^{d,v}(s \circ g) - \text{rk } H_0^{a,v}(s \circ g) + \text{rk } H_0^{c,v}(s \circ g) = 0.$$

- Set $h(p) = g^{-1} \circ f(p)$

# The rank invariant and uniqueness

**Theorem**

*Under reasonable conditions for $f, g$, for every $\varepsilon > 0$, a $\delta > 0$ exists such that if the matching distance between the functions $\rho_{s \circ f}(u, v)$ and $\rho_{s \circ g}$ is not greater than $\delta$ for every $s \in \Sigma_2$, then there exists a $C^1$-diffeomorphism $h : S^1 \to S^1$ such that $\|f - g \circ h\|_\infty \leq \varepsilon$.*

# Conclusions

Persistence modules might not be the most suitable tool for comparing geometric objects because

- on one hand, there are more morphisms between persistence modules than only those induced from the geometric category, even for curves
- on the other hand, considering only the rank invariant but in multidimensional persistence is sufficient to reconstruct curves

# Acknowledgments

- Cagliari, Ferri, Gualandri, L., Persistence Modules, Shape Description, and Completeness, Proc. CTIC 2012, LNCS 7309, 148-156.
- Frosini, L., Uniqueness of models in persistent homology: the case of curves, INVERSE PROBLEMS, Vol 27:12

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THANK YOU FOR YOUR ATTENTION!