Homology and complexity

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Introduction

Combinatorial algebraic topology:

- Given an interesting class of combinatorial objects (graphs, posets, lattices, hyperspace arrangements...)
- ... and a construction which assigns to them topological spaces ...
- ... what are the relations between the combinatorial and topological properties?

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Complexity theory

- Given a class of spaces...
- ... how efficiently can we compute their topological invariants?

Homology

Question

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$$H_*(L) \quad ?$$

Find easy/hard instances for this problem.

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$$H_*(L) \quad ?$$

Find easy/hard instances for this problem.

$$L \quad \to \quad C_*(L) \quad \to \quad H_*(L)$$

- list of all faces — poly-time
- list of facets

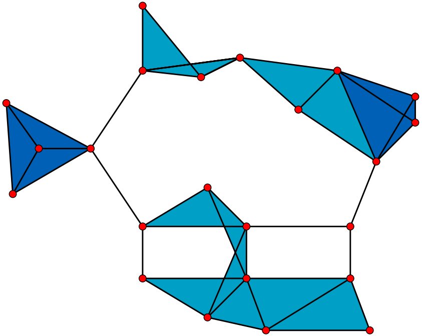
$$2^{\dim(L)} \cdot (\#\text{facets})$$

- 1-skeleton of a flag complex

$$2^{(\#\text{vertices})}$$

- $H_k(L)$ for a fixed dimension k — poly-time

- If G is a graph, then the clique complex $\mathrm{Cl}(G)$ is the simplicial complex whose faces are the cliques (complete subgraphs) of G.



Source: Wikipedia

- Geometric example: Vietoris-Rips complexes.
- a.k.a. flag complexes

Independence complexes

- If G is a graph, then the independence complex $\mathrm{Ind}(G)$ is the simplicial complex whose faces are the independent sets (edge-free sets) of G.
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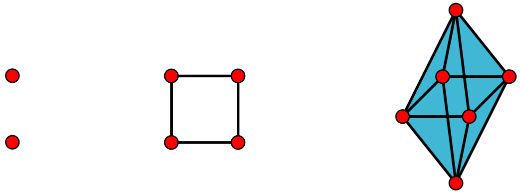
- Example: cross-polytope boundaries

$\mathrm{Ind}(e)$ $\qquad$ $\mathrm{Ind}(e \sqcup e)$ $\qquad$ $\mathrm{Ind}(e \sqcup e \sqcup e)$

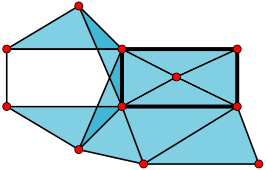


S^0 $\qquad$ $S^1 = S^0 * S^0$ $\qquad$ $S^2 = S^0 * S^0 * S^0$

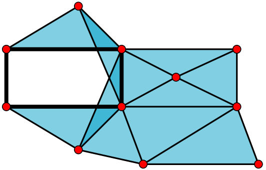
Spherical homology classes

- The "smallest" homology classes in flag complexes are given by embedded cross-polytopal spheres:

$$S^{k-1} = \underbrace{S^0 * \cdots * S^0}_{k} \hookrightarrow L \text{ gives an element of } \mathrm{H}_{k-1}(L).$$



trivial



non-trivial

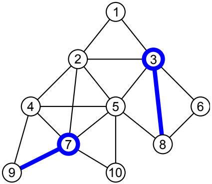
Cross-cycles and matchings

Definition (Cross-cycle, topologically)

A cross-cycle in a flag complex L is a homology class defined by an embedded subcomplex $S^0 * \cdots * S^0$, such that at least one face of that subcomplex is a maximal face of L. Such a homology class is non-zero.

Definition (Cross-cycle, combinatorially)

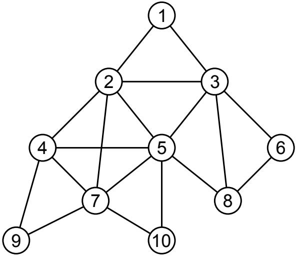
A cross-cycle in $\mathrm{Ind}(G)$ corresponds to an induced matching in G which contains a maximal independent set of G.

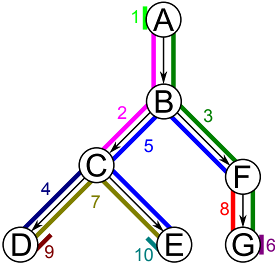


Defines an element of $\mathrm{H}_1(\mathrm{Ind}(G))$.

Chordal graphs

- A graph is chordal iff it is an intersection graph of subtrees of a fixed tree.





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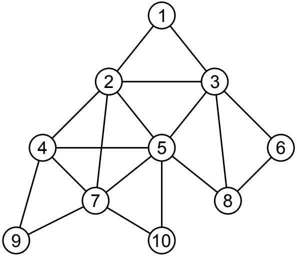
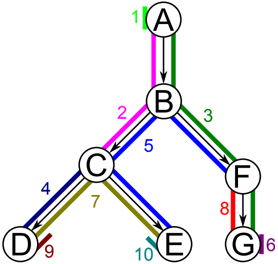




- Extensively studied in algorithmic graph theory.
- Well studied $\mathrm{Ind}(G)$: topologically (Woodroofe, Kawamura, Engström, ...), algebraically (van Tuyl, Engström, Dao).

Topology vs. combinatorics

Theorem (Homology of chordal graphs is combinatorializable)

If G is a chordal graph then the homology group $\mathrm{H}_*(\mathrm{Ind}(G))$ has a basis consisting of cross-cycles.

Consequently,

- $\mathrm{H}_{k-1}(\mathrm{Ind}(G))$ *is non-trivial*

if and only if

- *G has an induced matching of size k containing a maximal independent set.*

Main results

Theorem (Poly-time)

The following problem is solvable in polynomial time.

- *Given a chordal graph G, decide if $\mathrm{Ind}(G)$ has trivial homology in all dimensions:*

$$\widetilde{H}_i(\mathrm{Ind}(G)) = 0 \text{ for all } i \text{ ?}$$

Theorem (Hardness)

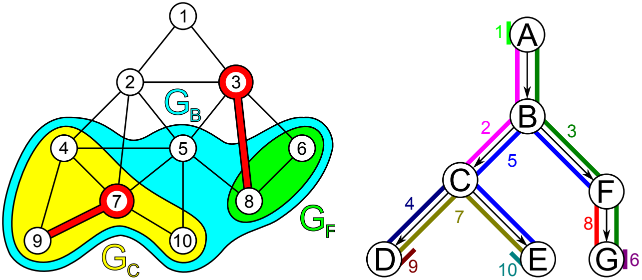
The following problem is NP-complete.

- *Given a chordal graph G and an integer k, decide if*

$$\mathrm{H}_k(\mathrm{Ind}(G)) = 0 \text{ ?}$$

The algorithm for chordal graphs

- A bottom-up recursion in the tree model of G.



- Computes special solutions that can be easily merged over subtrees.
- But no control over size.

Hardness consequences

Theorem (Homology is hard for flag complexes)

The following problem is NP-hard:

- *Given a graph G and an integer k, decide if*

$$\mathrm{H}_k(\mathrm{Cl}(G)) = 0 \quad ?$$

Theorem (Homology is hard for arbitrary simplicial complexes)

The following problem is NP-hard:

- *Given a simplicial complex L, presented as the list of maximal faces (facets), and an integer k, decide if*

$$\mathrm{H}_k(L) = 0 \quad ?$$

[Answers V. Kaibel, M. E. Pfetsch, *Some Algorithmic Problems in Polytope Theory*]

Questions / other work

- Connectivity $\geq k$?

$$\mathrm{H}_i(L) = 0 \quad \text{for} \quad i \leq k.$$

- Homological dimension $\leq k$?

$$\mathrm{H}_i(L) = 0 \quad \text{for} \quad i > k.$$

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- B. H. Roune, E. Sáenz de Cabezón, `arxiv/1112.4523`, Computing Euler characteristic is #P-complete.

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Thank you!