

# Multiple Dirichlet Series and Applications to Automorphic Forms

## WORKSHOP REPORT

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### 1. Relation to the original proposal

There were no significant deviations from the proposal.

### 2. Overview

Zeta and  $L$ -functions are fundamental objects in Number Theory and their deeper understanding is crucial for most major programmes in the subject. The study of multiple Dirichlet series (MDS) began more recently than that of classical Dirichlet series but is rapidly developing and, in recent years, there has been a surge of important breakthroughs. The main achievements in the theory so far include the construction of Weyl group multiple Dirichlet series, applications of MDS to the problem of non-vanishing of twists of  $L$ -functions, to convexity breaking, to moments of zeta and  $L$ -functions and the connections to multiple zeta function values. The workshop brought together the main workers in the field of MDS and other leading international experts in allied subjects, as well as younger researchers and post-graduate students working in related areas. The workshop centred around excellent talks reporting on the most recent developments in MDS and in areas that have the potential of fruitful interaction with the subject. We expect that the questions during the talks and the one-to-one discussions during the meeting will stimulate further progress in the major problems and will point to further research directions in the broader area of MDS.

### 3. Comprehensive report

The meeting was attended by 69 participants based in 13 countries (US 29, UK 15, Germany 6, Israel 5, Japan 5, France 2, Canada 1, Denmark 1, India 1, Italy 1, Korea 1, Switzerland 1, the Netherlands 1), including 10 postgraduate students. The list of participants included leading experts as well as younger researchers working on the theory of MDS (including their applications to problems of non-vanishing problems for  $L$ -functions, to integral moments, unitary periods etc.), automorphic representation theory, Langlands programme, trace formulas, multiple zeta functions, quasi-modular forms, multiple zeta values, singular moduli, mean-periodicity and other analytic aspects of automorphic  $L$ -functions.

The main axes around which the workshop revolved were two short courses. In the first, *WMD's and Crystals*, the speakers (Solomon Friedberg, Ben Brubaker, Gautam Chinta, Paul Gunnells, Daniel Bump) reported on their recent striking results on new methods of constructing Weyl group MDS, on the  $p$ -parts of the coefficients of a  $n$ -th order MDS, on the interpretation of these coefficients as metaplectic Whittaker coefficients etc.

In the second short course, *Integral Moments*, important applications of MDS to the study of integral moments (sub-convexity bounds of  $L$ -functions over number fields in the  $t$ -aspect, the analytic continuation of the fourth moment MDS over function fields, asymptotic formulas etc.) were discussed by the researchers who established them (Paul Garrett, Adrian Diaconu, Alina Bucur).

These two courses provided the framework for reports on further recent work related to MDS and their applications:

S. Frechette presented a construction of a Weyl group MDS associated to the root system  $B_n$  and a proof of their functional equations for  $n=1$ .

P. Garrett discussed spectral identities that involve second moments of  $L$ -functions.

A. Goncharov analyzed arithmetic aspects of multiple zeta values and their interpretation in terms of modular varieties.

A. Kontorovich showed how non-vanishing results on the first non-vanishing quadratic twist of  $L$ -series on  $GL(r)$  ( $r=1, 2, 3$ ) can be obtained using MDS.

O. Offen's talk dealt with period integrals over unitary groups and an application of Jacquet's factorization process in this context.

K. Matsumoto outlined an analytic study of multiple zeta-functions associated to root systems, including analytic continuation, functional equations and the location of poles.

S J Patterson discussed arithmetic aspects of coefficients of metaplectic MDS and a possible Hecke-theoretic interpretation.

These talks were complemented by presentations on research in related questions which have the potential of pointing to future research directions in the theory of MDS:

Yj Choie discussed connections between Jacobi-like forms and quasi-modular forms and outlined a theory of Hecke operators for quasi-modular polynomials.

B. Heim presented a trace formula that relates special values of Hecke  $L$ -functions to Rankin  $L$ -functions and the Garrett triple  $L$ -function.

J. Hundley's talk focused on the extension of a Rankin-Selberg integral construction carried out by Ginzburg to the quasi-split case.

O. Imamoglu discussed recent results on cycle integrals of  $j$  that are analogous to earlier results by Borcherds and Zagier linking traces of singular moduli and weakly holomorphic modular forms of half-integral weight.

W. Luo presented results about the asymptotic variance of the distribution of closed geodesics on the modular surface

M. Suzuki discussed a new connection between a certain class of complex functions closed under multiplication and division and the classes of mean-periodic functions.

**Outcomes:** The main objectives of the workshop were:

- To present mini-courses on the foundations and recent research in the theory of MDS.
- To formulate a general programme for future development of the subject and present the main outstanding conjectures.
- To discuss and outline the most promising research directions for future breakthroughs in the subject.
- Via active participation of the UK and EU number theorists in the workshop to stimulate further development of the subject.

The actual workshop and the formal and informal responses of the participants to it show that these objectives were met. Specifically,

- In the two short courses outlined above, the foundations of the theory of MDS were described and recent developments in the theory and its applications were presented.
- The outstanding questions of the subject were formulated in the two courses and the other talks by the main workers in the area of MDS. That included extensions of already proved results on Weyl group MDS and on integral moments to more general contexts, the possibility of additional results on non-vanishing of  $L$ -functions using MDS, further arithmetic information about period integrals over a unitary group and about metaplectic MDS etc.
- In addition to the precise delineation of the context and the formulation of the outstanding questions in the area, the main researchers working on MDS discussed extensively work in progress and possible further research directions towards these conjectures. At the same time, the lectures on related subjects suggested promising research directions that originate in other areas, as indicated for instance in D. Kazhdan's, .A. Goncharov's, and K. Matsumoto's talks. Some of these directions were discussed both during the talks and in informal one-to-one discussions.
- The workshop was attended by a large number of researchers based in the UK and EU and this will further stimulate the development of the subject. According to the responses to the official workshop questionnaire, the workshop has already led to possible fruitful collaborations. This is particularly important as that was the first major activity on MDS to be held outside the US and it should lead to work on MDS in a broader geographic context.

In summary, the success of the workshop exceeded the expectations of its organizers. The original requests for participation and the actual attendance numbers were higher than predicted. In fact, as a result, the workshop was slightly over budget, but that was managed by ICMS. A large number of high profile number-theorists attended and actively contributed to the talks. The aim of attracting the interest of UK number theorists was achieved, with essentially all researchers working in the area to which MDS belongs participating. The number of post-graduate students (10) as well as women participants (11) and speakers (4) was also unusually high.

The mathematical level of the workshop was exceptionally high. All scientific objectives were achieved and the impact of the workshop in terms of interaction between MDS and other areas and in terms of emerging collaborations has already started manifesting itself.

The scientific and administrative organization was very successful. We thank the ICMS staff, and especially Irene Moore and Johanna Kytöharju for their excellent work both during the preparation period and through the course of the meeting.