EFFECTIVE REAL ANALYTIC GEOMETRY
WORKSHOP REPORT

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1. Relation to the original proposal

There were no significant deviations from the proposal.

2. Overview

Effective real analytic geometry has its origins in several sources. First, in the theory of semi- and subanalytic sets and their complements founded in the late 60-s, and since developed further by many authors. It was suggested that the o-minimal structure of the classes of semialgebraic or subanalytic sets makes precise the Grothendieck's vision of a "tame topology". O-minimality links subanalytic geometry to modern applied model theory. Model-theoretic methods allowed to obtain the first proofs of complement theorems (model completeness, in the language of logic) for some classes of real analytic functions, which were unassailable by geometric techniques of the previous stage. Another origin of effective analytic geometry lies in the theory of *fewnomials*, with its underlying ideology that the geometric objects described by "short" formulae should have a "simple" topology. Finally, very significant progress was achieved in the semialgebraic case, where the defining functions are polynomials, in particular in Thom-Milnor style upper bounds on the homological complexity of semialgebraic sets, and in computational real algebraic geometry. All these strands are now becoming very close to one another, which helps to produce significant new results. The workshop brought together leading international experts, as well as younger researchers, in real analytic geometry, o-minimality, algebraic computational complexity, and related areas of topology and model theory. Many excellent talks reported on the most recent developments. We expect that the lively discussions that followed will result in a noticeable progress in the current outstanding problems, and will stimulate the younger researches to study and apply the variety of the available mathematical techniques.

3. Extended report

The meeting had 33 participants from 6 countries (UK 10, France 9, USA 7, Canada 4, Israel 2, Germany 1), including 10 younger researchers (UK 6). Within the topic of the conference, there was a wide range of interests among the participants, spreading over algebraic geometry, singularity theory, model theory, dynamical systems, topology, and computational complexity. Prof Andrei Gabrielov, to whom the meeting was dedicated on the occasion of this 60th birthday, made fundamental contributions to all these areas.

There were 22 hour-long talks. Several themes were represented.

O-minimality. Gareth Jones reported new model completeness results for polynomially bounded o-minimal expansions of the reals. He applied this to certain exponential structures. **Leonard Lipshitz** presented a general framework for studying real closed fields which supports a class of analytic functions (in addition to the usual algebraic functions). **Angus Macintyre** talked on decidability of the theory of Weierstrass elliptic functions under the assumption of Andre's conjecture. **Jean-Philippe Rolin** proved that there exists an o-minimal structure without the analytic cell decomposition property. **Patrick Speissegger** reported on a new exciting development connecting model theory and dynamical systems, showing the o-minimality of the structure generated by all transition maps near non-resonant hyperbolic singularities.

Singularity theory and effective resolution of singularities. Edward Bierstone presented a comprehensive survey of the current state of the algorithmic problems in resolution of singularities on characteristic 0, an important tool in subanalytic geometry, which is becoming useful in establishing o-minimality of certain structures related to dynamical systems (e.g., as in Speissegger's talk). **Vincent Grandjean** explained a result about the asymptotic of the length of a geodesic, approaching an real algebraic surface singularity. **Victor Palamodov** talked about a proof of a weaker form of the conjecture (supported by extensive computer calculations), that there is an isomorphism between the algebra of of the modular stratum of a non-contractible hypersurface germ and its Milnor algebra. **Bernd Martin** developed this topic further, and showed how computer calculations prove the conjecture for all unimodal singularities. **Adam Parusinski** presented complete characterizations of blow-analytic equivalence for two variable real analytic function gems in terms of their minimal resolution and their real tree models. **David Trotman** described recent advances in bi-Lipschitz stratifications of definable sets via triangulations which include Valette's proof of a well-known conjectire of Siebenmann and Sullivan (1977).

Topological bounds and enumerative geometry. Saugata Basu presented a new o-minimal framework leading to some far-reaching generalizations of upper bounds on combinatorial and topological complexity of subspace (or non-linear) arrangements. His technique makes use of some of the recent advances in the upper bounds on Betti numbers and the number of homotopy types of definable sets, due to himself and Gabrielov et al. **Frank Sottile** explained his recent results with Bihan on a significant improvement of the classical Khovanskii's upper bound on the number of isolated solutions of a non-degenerate system of *fewnomial* equations over the reals. Sottile-Bihan bound is asymptotically optimal and is based on a technique of Gale dual systems. **Frederic Bihan** showed how, basing on this bound, to obtain new upper bounds on the sum of Betti numbers of a fewnomial hypersurface. **Ilia Itenberg** gave a survey of the lower bounds on the number of real rational curves passing through a given real generic collection of points on a real rational surface. The main tools here include Mikhalkin's correspondence between real and tropical objects, and Welschinger invariants. **Dmitry Novikov** considered upper bounds on the number of limit cycles appearing in perturbations of Darboux integrable vector fields. Unlike previously known boundedness results for polynomial Hamilton vector fields, the algebraic properties are non longer valid, and tools of more analytic nature are employed. **Thierry Zell** described a range of

applications of the "descent" spectral sequence to new upper bounds on topological complexity of sets definable with quantifiers.

Semi-algebraic and sub-analytic geometry. Andrei Gabrielov explained the recent progress in the study of Eigenfunctions of the Schrödinger operator on the real line with a polynomial potential. It turns out that Eigenfunctions are associated with certain infinite planar trees ("Dessins d'Enfants"), and the braid group action on the trees relates the Eigenfunctions to the coefficients of the potential. **Viacheslav Kharlamov** talked about the *chirality* problem: which cubics are not deformation equivalent to their image under a mirror reflection. The answer is given in terms of the Eigen-sublattices of the complex conjugation involution in homology, and is effectively applied to concrete examples. **Askold Khovanskii** presented an explicit formula, based on polynomial continuous fractions, to compute a degree of a rational map of the real projective space to itself. This implies simple and effective proofs of Sturm's and Tarski's theorems. Moreover, the formula provides an important algorithmic subroutine which can be used as a crucial ingredient in a singly exponential algorithm for quantifier elimination over real closed fields and over algebraically closed fields. **Krysztof Kurdyka** reported new results about trajectories of "horizontal" gradients of polynomials, with respect to the sub-Riemannian metric. He showed that the Łojasiewicz gradient inequality does not hold in this setting but, for generic polynomials, the trajectories approaching the critical set have limits.

Outcomes. The main aim of the workshop was to bring together leading international experts in real analytic geometry, o-minimality, algebraic computational complexity, and related areas of topology, model theory, and theoretical computer science, to share concepts and techniques developed in these areas. This aim was successfully achieved: world-leading figures in each subject lectured at the workshop, informal discussions were running permanently. The meeting provided an overview of the current state of research in real analytic geometry with the emphasis on "tameness" and effectiveness.

As the numbers show, there was a significant proportion of younger participants, postgraduate students and postdocs, and at all times they were encouraged to interact and participate in the conference. The feedback that we have from many of them shows their high appreciation of the possibility to learn new ideas and to have personal in-depth discussions with the leading experts in the field. The talk by Askold Khovanskii was highlighted as particularly important by several younger researchers.

A number of participants (Rolin, Grandjean, Parusinski, Kirby, Sottile, Jones) indicated that communications and discussions at the meeting are likely to produce new ideas for future research. In particular, Gareth Jones reported that he proved a new result using the ideas from the meeting, which will be the basis of his new research paper. Obviously, in most other cases the concrete outcomes may show themselves somewhat later.

The participants mark the high mathematical level of the workshop, and seem to be very satisfied with its scientific organization and general administration. We join in the prase of the latter: the work of ICMS staff, in particular of Irene Moore, was superb, both during the preparation period and through the course of the meeting.