

# Report on the ICMS workshop: The Riemann-Hilbert Problem and Toeplitz Operators

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## Scientific Organisers:

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## Short description of the meeting

Applications of the Riemann-Hilbert problem to integrable systems, orthogonal polynomials and random matrices is one of the most vigorously developing branches of mathematics at present. The recent research in this area has resulted in spectacular progress in many directions, especially those related to the study of asymptotic behaviour of integrable models, and in creation of new powerful methods such as the nonlinear steepest descent method for the Riemann-Hilbert problem.

The spectral theory of Toeplitz operators has also seen significant recent progress in several directions including operators with piece-wise continuous symbols on general Carleson curves with arbitrary Muckenhoupt weights and operators with matrix semi-almost periodic symbols. There is also quite a number of recent striking, often counterintuitive, results on Toeplitz matrices. The latter is a subject attracting a great deal of interest from researchers involved with numerical linear algebra, pseudospectra,  $C^*$ -algebras, orthogonal polynomials, random matrices, representation theory, stochastic processes and other fields.

The main aim of the workshop was to bring together the top researchers who apply the Riemann-Hilbert problem to integrable systems, orthogonal polynomials and random matrices, and those who work in the spectral theory of Toeplitz operators, to review the current state of the field and, most importantly, to discuss plans for the future.

The workshop was attended by 34 researches representing a wide variety of backgrounds and interests. The programme consisted of six 50 minutes long talks and twenty 30 minutes long talks. A most important feature of the timetable were very long lunch breaks from 12:30–12:45 to 15:50. These breaks provided the participants with plenty of time for discussions and much mathematics was done during these periods.

## Scientific Content

Below, we give a very brief description of the topics covered by the talks.

*Jinho Baik* presented his joint results with R. Buckingham, J. DiFranco and A. Its on computation of total integrals of real and purely imaginary solutions to Painlevé II equations on the real line.

*Estelle Basor* spoke on the asymptotics of a block Toeplitz determinant which arises in the classical dimer model for the triangular lattice when considering the monomer-monomer correlation function.

*Yang Chen* spoke on a class of orthogonal polynomials that arises in the study of a singular linear statistics motivated by problems in integrable quantum field theory, and whose recurrence coefficients are related to a Painleve III equation.

*Maurice Duits* presented his joint results with A. Kuijlaars on characterisation of the limiting measure of the eigenvalue distribution of banded Toeplitz matrices in terms of an equilibrium problem.

*Torsten Ehrhardt* spoke on gap probabilities in random matrix theory and the asymptotics of Wiener-Hopf plus Hankel determinants with Fisher-Hartwig symbols.

*Miroslav Engliš* discussed connections of Toeplitz operators with various fields of mathematics ranging from quantization to group representations and function theory in several complex variables.

*Thanasis Fokas* presented a new method for analytic inversion of certain integral transforms which involves the Riemann-Hilbert and  $\bar{\partial}$ -problems and which can be applied in a wide variety of fields including imaging and initial boundary value problems with a moving boundary.

*Sergei Grudsky* presented his joint results with A. Böttcher on the boundedness of Toeplitz type matrices with variable coefficients.

*Alexander Its* gave a review of basic features, recent results and open questions in the asymptotic analysis of Toeplitz and Hankel determinants via the Riemann-Hilbert method including the fundamental connections to the theory of Painleve equations.

*Yuri Karlovich's* talk was devoted to boundedness, compactness and Fredholm properties of pseudodifferential operators with discontinuous symbols and to their applications to algebras of singular integral operators with shifts.

*Igor Krasovsky* spoke on the recent progress in calculation of the large  $n$  asymptotics for  $n$ -dimensional Hankel determinants with Fisher-Hartwig symbols.

*Arno Kuijlaars* spoke on the Riemann-Hilbert steepest descent analysis for non-intersecting squared Bessel paths.

*Marko Lindner* spoke on the Fredholm properties of operators from the Wiener algebra of infinite matrices whose entries are bounded linear operators acting in a Banach space.

*Mo Man Yue* presented his joint results with A. Its and F. Mezzadri on the asymptotics of the entropy of a generalized  $XY$  model and the connections with block Toeplitz determinants.

*Francesco Mezzadri* spoke on the study of the asymptotics of a Toeplitz determinant whose symbol has essential singularities approaching the unit circle and on the connection of this asymptotics to the properties of the zeros of the Riemann zeta-function and of its derivative.

*Jonathan Partington* presented his joint results with B. Atfeh, L. Baratchart and J. Leblond on the study of inverse problems for the Laplacian by means of an approximation problem in Hardy spaces, for which the solution is expressible in terms of Toeplitz operators.

*Vladimir Peller* spoke on perturbation of functions of contractions.

*Beatrice Pelloni* described the recent explicit solution of an initial boundary value problem with a moving boundary for linear evolution equations, obtained by casting the problem as a  $\bar{\partial}$ -equation in the complex plane.

*Sandra Pott* presented her joint results with B. Jacob and J.R. Partington on norm estimates for the problem of minimal-norm weighted tangential interpolation by functions in vector-valued Hardy spaces and on their applications to the controllability properties of linear semigroup systems with a Riesz basis of eigenvectors.

*Eugene Shargorodsky* gave a review of some recent progress and open problems in the study of the essential spectra of Toeplitz operators with bounded measurable coefficients.

*Ilya Spitkovsky* gave an overview of the currently known results on the existence and construction of the almost periodic factorization, and discussed related open prob-

lems.

*Nikolai Vasilevski* described the structure of the  $C^*$ -algebra generated by Toeplitz operators with piece-wise continuous symbols on the Bergman space.

*Jani Virtanen* spoke on boundedness, compactness and Fredholm properties of Toeplitz and Hankel operators on the Hardy space  $H^1$  and the Bergman space  $A^1$ .

*Elias Wegert* showed that any measurable function on the unit circle whose modulus is bounded by 1 admits an essentially unique “Blaschke representation”.

*David Wenzel* discussed a probability argument in favor of ignoring small singular values of matrices and illustrated the results by normal and multi-level Toeplitz matrices.

*Harold Widom* presented very sharp formulae for the inverse and determinant of a truncated Wiener-Hopf operator whose symbol has one double zero or two simple zeros.

## Conclusions

The success of the workshop owed much to the strong list of participants, the high standard of lectures by invited speakers and to the excellent environment at Heriot Watt which has encouraged many collaborative research projects. Many participants reported new collaborations or significant progress in on-going research resulted from the meeting. Here are some comments taken from the questionnaires completed by the participants:

“During the workshop I was able to collaborate with three other participants and to plan with them an approach to study several new problems.”

“Professor Basor and I are looking into a research project stimulated by one of the talks given in this workshop.”

“I want SPECIFICALLY single out the finalizing, during the workshop, of the joint project with A. Kuijlaars on the asymptotic analysis of the thirty forth Painleve equation which was originated during the previous ICMS workshop on Asymptotics and Modelling which took place in June of 2006 and which we both participated in.”

“I discussed with colleagues some problems and if we are successful that might result into papers, and perhaps open directions into new research in a broader way.”

“There is at least one participant with whom I had discussions that are likely to result in a joint paper eventually. There are two more with whom possible joint projects were tentatively discussed. The workshop also helped me revive/sustain my contacts with several other people.”

“I met with current collaborator. We are working on an interesting paper that is likely to be finished in the fall. We also developed new ideas for future work.”

The comments on the academic value of the workshop were very positive. Here are some of them:

“As far as the theory of Toeplitz operators is concerned, the workshop is the first attempt to bring together the experts from the theory of integrable systems and from more traditional directions of the operator theory. The level of participants was very impressive. The organizers succeeded in attracting practically all the top people in both fields who have been recently involved in the theory and application of Toeplitz operators. The overall academic value of the workshop was very high. Indeed, as a conference whose principal aim was to foster the integration of the methods of integrable systems into the mainstream of operator theory, the workshop is truly unique.”

“Very high, one of the best meeting I have attended in the last 10 years.”

“Excellent! High class people, very good talks, big success.”

“About half of the participants were leaders in the field of Riemann-Hilbert problems and Toeplitz operators. The workshop was really an event at the frontline of research

in the field. The talks were all extremely well prepared and presented, it was a true pleasure to listen the speakers.”

“I was very impressed at the excellence of the work of the people who were invited to give talks, and I would like to add the vast majority of these were superb.”

“The highlight for me was the opportunity to meet the experts in my field and to discuss my recent research with them. Meeting all the main figures in this field – they all seemed to be there.”

“This was an excellent opportunity to meet leading experts in the area, and to discuss and share results and ideas. A lot of informal discussions was one of the greatest virtue of the Workshop.”

There were only very few critical comments:

“I think it would have been even better if there had been some introductory talks, as the two themes were Riemann-Hilbert and Toeplitz, and I, for one, was rather unfamiliar with the first when we started.”

“I want to make though one critical remark. It would have considerably increased the effectiveness of the workshop if the organizers had put at the beginning a few expository lectures which would have summed up the main development in each of the two areas. Indeed ... it would be extremely beneficial to have review talks by A. Böttcher and P. Deift. The fact that the both were among the organizers should not have prevented (as apparently it had) from including them to the list of the speakers.”

“One comment on the schedule: the lunch break was about 3 1/2 hours. Although this was probably to allow informal discussion this seemed excessive to those with off-campus accommodations.”

The comments on facilities and accommodation were mostly positive, but there were the following negative ones:

“Not so impressed by my room in Heriot-Watt (with an obscenity written on the wall and not enough bedclothes), nor by the computing facilities (the advertised ones were closed, and it took a while to find something else suitable), but these are minor moans.”

“The only problem I had was that on some days/nights it was pretty cold in the rooms.”

“All were good except that the accommodation room was a little bit cold at night.”

It is a pleasure to thank EPSRC and ICMS for the financial support, and the staff of ICMS for their tireless, patient and invaluable help with all aspects of the organisation. The participants’ comments on the administrative arrangements were unanimously positive, with Johanna Kytöharju receiving particular praise.