Best constants in some Sobolev Poincaré trace inequalities

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Poincaré trace inequalities in BV

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If the boundary $\partial\Omega$ of Ω is smooth, then a linear operator is defined on the space $BV(\Omega)$ of functions of bounded variation in Ω, which associates with any function $u \in BV(\Omega)$ its (suitably defined) boundary trace $\widetilde{u} \in L^1(\partial\Omega)$.

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There exists a constant C, depending on Ω, such that

$$\inf_{t \in \mathbb{R}} \|\widetilde{u} - t\|_{L^1(\partial\Omega)} \leq C(\Omega)\|Du\|(\Omega)$$

for every $u \in BV(\Omega)$.

[MAZ'YA (1960)]

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[MAZ'YA (1960)]

We are interested in the minimization of $C(\Omega)$.

Poincaré trace inequalities in *BV*

We observe that the previous inequality is the case $p = 1$ (in *BV* setting) of the following one

$$\inf_{t\in\mathbb{R}} \|\widetilde{u} - t\|_{L^p(\partial\Omega)} \leq C_p(\Omega)\|Du\|_{L^p(\Omega)} \qquad (1)$$

where the extremal functions are solutions to the Stekloff eigenvalue problem

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ C_p(\Omega)|Du|^{p-2}\dfrac{\partial u}{\partial \nu} = |u|^{p-2}u & \text{on } \partial\Omega. \end{cases}$$

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The problem of minimizing the constant $C_p(\Omega)$ in (1) has been solved only for $p = 2$. The Weinstock-Brock inequality asserts that the (unique) minimizer among sets with fixed measure is the ball.

[Weinstock (1954)], [Brock (2001)]

Poincaré trace inequalities in *BV*

A property of L^1 norms ensures that the infimum

$$\inf_{t\in\mathbb{R}} \|\widetilde{u} - t\|_{L^1(\partial\Omega)}$$

is attained when t agrees with the median of $\widetilde{u}$ on $\partial\Omega$, given by

$$\mathrm{med}_{\partial\Omega}\widetilde{u} = \sup\{t \in \mathbb{R} : \mathcal{H}^{n-1}(\{\widetilde{u} > t\}) > \mathcal{H}^{n-1}(\partial\Omega)/2\}$$

[CIANCHI - PICK (2003)]

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Thus, the trace inequality is equivalent to

$$\|\widetilde{u} - \mathrm{med}_{\partial\Omega}\widetilde{u}\|_{L^1(\partial\Omega)} \le C_{\mathrm{med}}(\Omega)\|Du\|(\Omega)$$

for every $u \in BV(\Omega)$, where $C_{\mathrm{med}}(\Omega)$) denotes the optimal – smallest possible – constant for which the inequality holds true.

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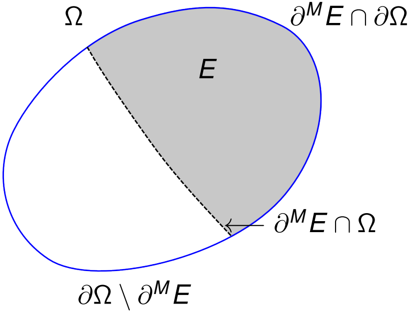
The constant $C_{\rm med}(\Omega)$ can be characterized as a genuinely geometric quantity associated with Ω, namely,

$$C_{\rm med}(\Omega)) = \sup_{E\subset\Omega} \frac{\min\{\mathcal{H}^{n-1}(\partial^M E\cap\partial\Omega)\,,\mathcal{H}^{n-1}(\partial\Omega\setminus\partial^M E)\}}{\mathcal{H}^{n-1}(\partial^M E\cap\Omega)},$$

where the supremum is extended over all measurable sets $E\subset\Omega$ with positive Lebesgue measure and $\partial^M E$ denotes the essential boundary of E.

[MAZ'YA (1968)]



Poincaré trace inequalities in BV

Theorem ([CIANCHI - FERONE - NITSCH - T., Crelle (to appear)])

We have:

$$C_{med}(\Omega) \geq \sqrt{\pi}\,\frac{n}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = C_{med}(B). \tag{2}$$

Moreover, equality holds in (2) *if and only if* Ω *is equivalent to a ball, up to a set of* $\mathcal{H}^{n-1}$ *measure zero.*

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Remark. The constant which appears in (2) coincides with

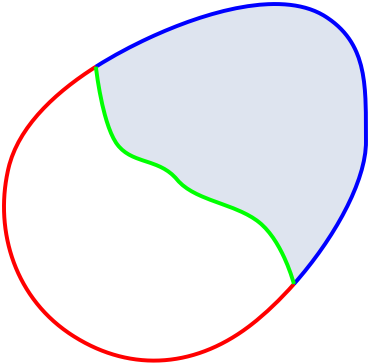
$$\frac{n\omega_n}{2\omega_{n-1}},$$

where $\omega_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The supremum which defines $C_{\mathrm{med}}(B)$ is attained at a half-ball.

[Burago-Maz'ya (1969)], [Bokowski - Sperner (1979)], [Escobar (1999)]

Poincaré trace inequalities in *BV*

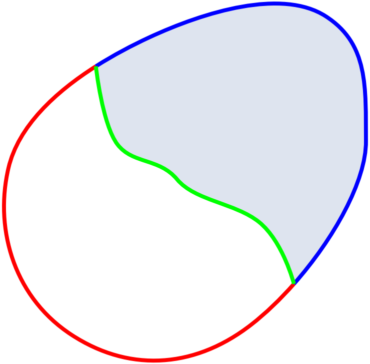
$$C_{\text{med}}(\Omega) = \sup_{E \subset \Omega} \frac{\min\{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega), \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)\}}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}$$



Poincaré trace inequalities in *BV*

When Blue and Red have the same $\mathcal{H}^{n-1}$ measure, i.e. $Per(\Omega)/2$ then

$$C_{\mathrm{med}}(\Omega) \geq \frac{Per(\Omega)}{2} \frac{1}{\displaystyle\inf_{\substack{H \text{ half-space} \\ \mathcal{H}^{n-1}(\partial^M(H\cap\Omega)\cap\partial\Omega)=\frac{Per(\Omega)}{2}}} \mathcal{H}^{n-1}(\partial^M(H\cap\Omega)\cap\Omega)}$$



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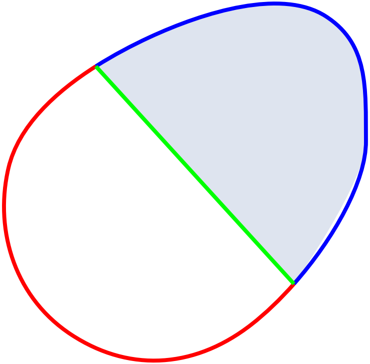

Cauchy surface area formula for convex sets

In order to prove the theorem in the case Ω is convex, it is possible to use the following Cauchy surface area formula:

$$Per(\Omega) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(\Pi_\nu \Omega)\, d\nu,$$

where $\Pi_\nu \Omega$ denotes the projection of Ω on the hyperplane with normal vector $\nu \in \mathbb{S}^{n-1}$.

[BURAGO - ZALGALLER (1988)]

An approach to the proof of the first result

We have to estimate from below the quantity

$$C_{\text{med}}(\Omega) = \sup_{E \subset \Omega} \frac{\min\{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)\,,\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)\}}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}.$$

On the other hand we have

$$\begin{aligned}
\min_{\substack{H \text{ half-space} \\ \mathcal{H}^{n-1}(\partial^M (H\cap\Omega)\cap\partial\Omega) = \frac{Per(\Omega)}{2}}} \mathcal{H}^{n-1}(\partial^M (H \cap \Omega) \cap \Omega) &\le \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} h(\nu)\, d\nu \le \\
&\le \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(\Pi_\nu \Omega)\, d\nu,
\end{aligned}$$

where

$$\mathcal{H}^{n-1}(\partial^M (H_\nu \cap \Omega) \cap \partial\Omega) = \frac{Per(\Omega)}{2}, h(\nu) = \mathcal{H}^{n-1}(\partial^M (H_\nu \cap \Omega) \cap \Omega)$$

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$$C_{\text{med}}(\Omega) \geq \frac{Per(\Omega)}{2} \frac{1}{\displaystyle\min_{\substack{H \text{ half-space} \\ \mathcal{H}^{n-1}(\partial^M(H\cap\Omega)\cap\partial\Omega)=\frac{Per(\Omega)}{2}}} \mathcal{H}^{n-1}(\partial^M(H\cap\Omega)\cap\Omega)}$$

using Cauchy formula, we have:

$$C_{\text{med}}(\Omega) \geq \frac{Per(\Omega)}{2} \frac{1}{\displaystyle\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}} \mathcal{H}^{n-1}(\Pi_\nu\Omega)\,d\nu} \frac{n\omega_n}{\omega_{n-1}} = \frac{n\omega_n}{2\omega_{n-1}} = C_{\text{med}}(B)$$

and the proof of the inequality is complete.

An approach to the proof of the first result

If equality holds in the previous inequality, that is,

$$C_{\text{med}}(\Omega) = \sup_{E \subset \Omega} \frac{\min\{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega), \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)\}}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)} = \frac{n\omega_n}{2\omega_{n-1}},$$

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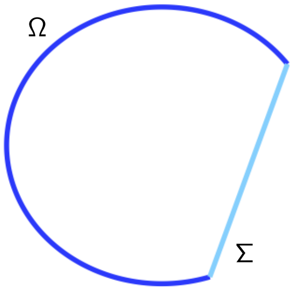
all the inequalities used above hold as equalities and we have

$$\mathcal{H}^{n-1}(\partial^M(H_\nu \cap \Omega) \cap \Omega) = \mathcal{H}^{n-1}(\Pi_\nu(\Omega)) = Per(\Omega)\frac{\omega_{n-1}}{n\omega_n}, \qquad \forall \nu \in \mathbb{S}^{n-1}.$$

It follows that Ω is, in fact, strictly convex. Indeed, assume, by contradiction, that there exist a straight line intersecting $\partial\Omega$ in a whole segment Σ.



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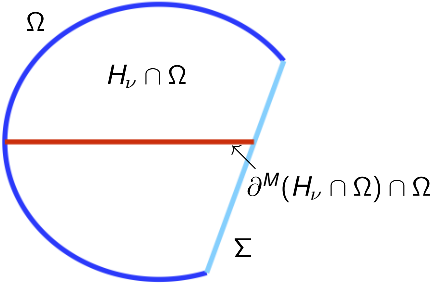
$$\mathcal{H}^{n-1}(\partial^M(H_\nu \cap \Omega) \cap \Omega) < \mathcal{H}^{n-1}(\Pi_\nu(\Omega)).$$



An approach to the proof of the first result

By the strict convexity of Ω, we have:

$$\mathcal{H}^{n-1}(I_\nu(\Omega)) = \mathcal{H}^{n-1}(\partial\Omega \cap H_\nu) = Per(\Omega)/2, \qquad \forall \nu \in \mathbb{S}^{n-1},$$

where $I_\nu(\Omega)$ denotes the *illuminated portion* of Ω. In particular,

$$\mathcal{H}^{n-1}(I_\nu(\Omega)) = \mathcal{H}^{n-1}(I_{-\nu}(\Omega)), \qquad \forall \nu \in \mathbb{S}^{n-1}.$$

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Finally, on calling B the ball with the same perimeter as Ω, we infer that

$$\mathcal{H}^{n-1}(\Pi_\nu(\Omega)) = \mathcal{H}^{n-1}(\Pi_\nu(B)), \qquad \forall \nu \in \mathbb{S}^{n-1}.$$

Hence, we conclude that Ω is a ball.

(The last two assertions come from well known results about convex bodies [GROEMER (1996)])

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A stronger version of Poincaré trace inequality holds, when $\mathrm{med}_{\partial\Omega}\widetilde{u}$ is replaced with the mean value $\widetilde{u}_{\partial\Omega}$ of $\widetilde{u}$ over $\partial\Omega$, defined as

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Observe that one has

$$C_{\mathrm{med}}(\Omega) \leq C_{\mathrm{mv}}(\Omega)$$

for every domain Ω.

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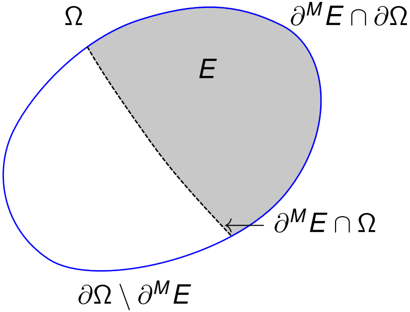
The constant $C_{\text{mv}}(\Omega)$ can be characterized as a genuinely geometric quantity associated with Ω, namely,

$$C_{\text{mv}}(\Omega) = \frac{2}{\mathcal{H}^{n-1}(\partial\Omega)} \sup_{E\subset\Omega} \frac{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)\, \mathcal{H}^{n-1}(\partial\Omega \setminus \partial^M E)}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)},$$

where the supremum is extended over all measurable sets $E \subset \Omega$ with positive Lebesgue measure.

[CIANCHI (2012)]



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Theorem ([Cianchi - Ferone - Nitsch - T., Crelle (to appear)])

If $n \geq 3$, *then*

$$C_{mv}(\Omega) \geq \sqrt{\pi}\,\frac{n}{2}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} = C_{mv}(B), \tag{3}$$

and the equality holds in (3) *if and only if* Ω *is equivalent to a ball, up to a set of* $\mathcal{H}^{n-1}$ *measure zero.*
If $n = 2$, *then*

$$C_{mv}(\Omega) \geq 2 = C_{mv}(B), \tag{4}$$

and the equality holds in (4) *if* Ω *is a disc. However there exist open sets* Ω, *that are not equivalent to a disc, for which equality yet holds in* (4).

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and the equality holds in (4) *if Ω is a disc. However there exist open sets Ω, that are not equivalent to a disc, for which equality yet holds in* (4).

Remark. Also in this case the lower bounds which appear in (3) and (4) coincide with the values of $C_{\rm mv}$ computed on a ball. When $n \geq 3$, $C_{\rm mv}$ is attained at a half-ball, when $n = 2$, $C_{\rm mv}$ is attained in the limit, considering any sequence of circular segments whose measure converges to 0 (on the half-circle the ratio is $\pi/2$).

[Cianchi (2012)]

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When $n = 2$, inequality (5) still holds true, but the right-hand side does not coincide with the constant C_{mv} computed on a ball. Indeed,

$$\sqrt{\pi}\,\frac{n}{2}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}\Bigg|_{n=2} = \frac{\pi}{2} < 2.$$

An example

Let us consider a stadium-shaped domain

$$S_{R,d} = \text{convex hull of two discs of equal radii } R\text{, with centers at distance } d,$$

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In order to calculate $C_{\mathrm{mv}}(S_{R,d})$ one can reduce the analysis to subsets of $S_{R,d}$ bounded by a chord which is orthogonal to the flat parts of $S_{R,d}$.

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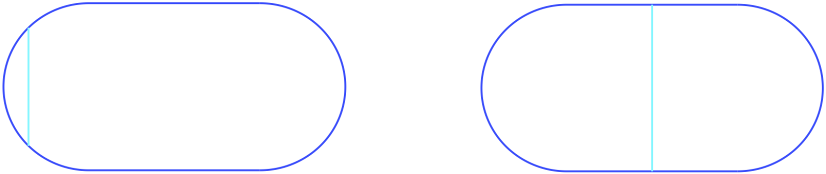

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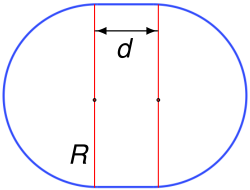
If $d \leq (4 - \pi)R$

$$C_{\text{mv}}(S_{R,d}) = 2.$$



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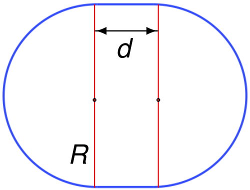
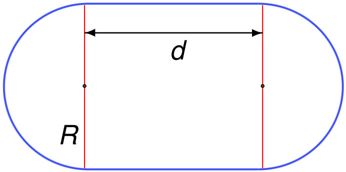


If $d > (4-\pi)R$

$$C_{\rm mv}(S_{R,d}) = \frac{d+\pi R}{2R} > 2.$$



Final remarks

Another stronger inequality can be stated using the median of u on Ω

$$\mathrm{med}_\Omega u = \sup\{t \in \mathbb{R} : |\{u > t\}| > |\Omega|/2\}.$$

More precisely there exists a constant $\bar{C}(\Omega)$, depending on Ω, such that

$$\inf_{c\in\mathbb{R}} \|\widetilde{u} - \mathrm{med}_\Omega u\|_{L^1(\partial\Omega)} \leq \bar{C}(\Omega)\|Du\|(\Omega) \tag{6}$$

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Also in this case the constant $\bar{C}(\Omega)$ can be expressed in terms of geometric quantities related to Ω, indeed the following characterization is known

$$\bar{C}(\Omega) = \sup_{\substack{E\subset\Omega \\ |E|\leq|\Omega|/2}} \frac{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}.$$

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What are the sets minimizing $\bar{C}(\Omega)$?

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$$\mathrm{med}_\Omega u = \sup\{t \in \mathbb{R} : |\{u > t\}| > |\Omega|/2\}.$$

More precisely there exists a constant $\bar{C}(\Omega)$, depending on Ω, such that

$$\inf_{c\in\mathbb{R}} \|\widetilde{u} - \mathrm{med}_\Omega u\|_{L^1(\partial\Omega)} \leq \bar{C}(\Omega)\|Du\|(\Omega) \tag{6}$$

for every $u \in BV(\Omega)$.
Also in this case the constant $\bar{C}(\Omega)$ can be expressed in terms of geometric quantities related to Ω, indeed the following characterization is known

$$\bar{C}(\Omega) = \sup_{\substack{E\subset\Omega \\ |E|\leq|\Omega|/2}} \frac{\mathcal{H}^{n-1}(\partial^M E \cap \partial\Omega)}{\mathcal{H}^{n-1}(\partial^M E \cap \Omega)}.$$

What are the sets minimizing $\bar{C}(\Omega)$? As usual, balls are the first candidates one looks for. Computing $\bar{C}(\Omega)$ on a ball will be the first step. $\bar{C}(B) \neq C_{\mathrm{med}}(B)$ since extremal functions in (6) are not simply the characteristic functions of half-ball.

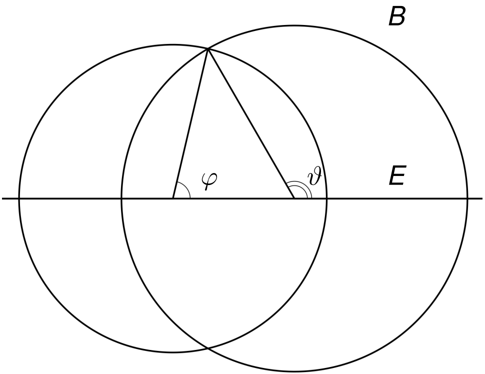

Figure : Intersection of B with a ball