Existence and regularity of solutions to optimal partition problems involving Laplacian eigenvalues

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Shape optimization and spectral geometry

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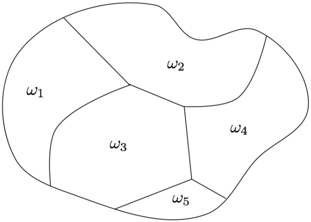
Outline

Outline

A Model Problem

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Take $m \in \mathbb{N}$ and $k_1, \ldots, k_m \in \mathbb{N}$. Denote $\lambda_{k_i}(\omega)$ as being the k_i–th eigenvalue of $(-\Delta, H^1_0(\omega))$.

$$\inf \left\{ \sum_{i=1}^{m} \lambda_{k_i}(\omega_i) : \ \omega_1, \ldots, \omega_m \subseteq \Omega \text{ open sets, } \omega_i \cap \omega_j = \emptyset \ \forall i \neq j \right\}$$



Goals:

- Existence of an optimal partition
- Regularity of the optimal partition
- Regularity of associated eigenfunctions
- Structure of the nodal set

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Optimal Partition Problems

- Class of admissible sets: $\mathcal{A}(\Omega)$
- Cost Functional: $\Phi : \mathcal{A}(\Omega)^m \to \mathbb{R}$

Minimization problem:

$$\inf \{\Phi(\omega_1, \dots, \omega_m) : \ \omega_i \in \mathcal{A}(\Omega), \ \omega_i \cap \omega_j = \emptyset \ \forall i \neq j\}$$

Applications:

- monotonicity formulas (Alt-Caffarelli-Friedman);
- nodal sets of eigenfunctions of Schrödinger operators;
- characterization of limits of elliptic systems with competitive interaction;
- inverse problems;

Optimal Partition Problems

The solvability of

$$\inf \{\Phi(\omega_1, \ldots, \omega_m) : \ \omega_i \in \mathcal{A}(\Omega), \ \omega_i \cap \omega_j = \emptyset \ \forall i \neq j\}$$

strongly depends on the choice of the class $\mathcal{A}(\Omega)$.

In general, for reasonable classes, such as open sets, such a problem does not admit a solution $\Longrightarrow$ a relaxation is needed.
[Butazzo and Dal Maso (1998)], [Buttazzo and Timofte (2002)]

Reference: The book *Variational methods in shape optimization problems* [Bucur and Buttazzo (2005)]

Outline

Basics on Sobolev capacity

$$\mathrm{cap}(K,\Omega) := \inf\{{}_{\Omega}|\nabla u|^2 \;:\; u \in H^1_0(\Omega), u \geq 1 \text{ a.e. on } K\}$$

- If a property $P(x)$ holds for all $x \in E$ except for the elements of a set Z of zero capacity, we say that $P(x)$ holds quasi-everywhere on E.
- A subset $A \subset \mathbb{R}^N$ is said to be quasi-open (resp. quasi-closed) if for every $\varepsilon > 0$ there exists an open (resp. closed) subset A_ε, such that $\mathrm{cap}(A_\varepsilon \Delta A) < \varepsilon$.
- A function $f : \Omega \to \mathbb{R}$ is said to be quasi-continuous. if for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : \Omega \to \mathbb{R}$ such that $\mathrm{cap}(\{f \neq f_\varepsilon\}) < \varepsilon$. It is well known that every function u of the Sobolev space $H^1(D)$ has a quasi-continuous representative, which is uniquely defined up to a set of capacity zero.

γ-convergence of sets

For a quasi open $A \subset \Omega$, we define w_A as the unique minimizer of the compliance problem:

$$\inf\{ \int_\Omega \frac{1}{2}|\nabla u|^2 - u \ : \ u \in H^1_0(\Omega)\, , u \equiv 0 \text{ on } A^c\} \, .$$

Definition (of weak γ-convergence)

We say that a sequence $(A_n)_n$ of A weakly γ-converges to A if $(w_{A_n})_n$ converges weakly in $H^1_0(\Omega)$ to a function $w \in H^1_0(\Omega)$ (that we may take quasi-continuous) such that $A = \{w > 0\}$.

Theorem (Compactness)

if Ω is bounded, then the weak γ-convergence on A is sequentially compact.

A general existence result by Bucur, Buttazzo, Henrot

Admissible sets: $\mathcal{A}(\Omega) = \{\omega \subset \Omega \text{ quasi open}\}$.

Theorem (Bucur, Buttazzo, Henrot 1998)

- *Φ is monotone nonincreasing with respect to domain inclusion;*
- *Φ is γ–weakly lower semicontinuous.*

Then the problem

$$\inf\left\{\Phi(\omega_1,\dots,\omega_m):\ \omega_i \subset \Omega \text{ quasi-open},\ cap(\omega_i \cap \omega_j) = 0\ \forall i \neq j\right\}.$$

admits a solution.

Example:

$$\Phi(\omega_1,\dots,\omega_k) = F(\lambda_{k_1}(\omega_1),\dots,\lambda_{k_m}(\omega_m)),$$

with $F : \mathbb{R}^m \to \mathbb{R}$ monotone nondecreasing and lower semicontinuous in each variable.

Problems

For a general

$$\Phi(\omega_1,\ldots,\omega_m) = F(\lambda_{k_1}(\omega_1),\ldots,\lambda_{k_m}(\omega_m)),$$

(model: $\Phi(\omega_1,\ldots,\omega_m) = \sum_{i=1}^{k} \lambda_{k_i}(\omega_i)$)

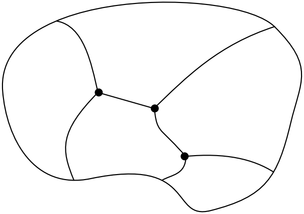
- Does the optimal partition admit an open representative?
- What about necessary (extremality) conditions? (needs assumptions on F)
  - first eigenvalues
  - higher eigenvalues
  - simple eigenvalues
  - multiple eigenvalues
- Approximation and penalization: phase separation for strongly coupled systems
- Further regularity of the partition?
  - open partitions
  - Structure of the nodal set (Federer's reduction and Almgren's stratification)
- Actual shape of the minimal partition

Outline

Definition of Regular Partition

Definition

An open partition $(\omega_1, \ldots, \omega_m)$ is called *regular* if:



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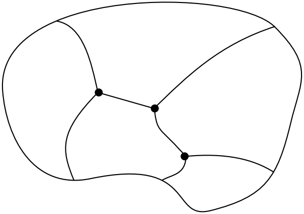
An open partition $(\omega_1, \ldots, \omega_m)$ is called *regular* if:

1. denoting $\Gamma = \Omega \setminus \bigcup_{i=1}^{m} \omega_i$, there holds $\mathscr{H}_{\dim}(\Gamma) \leq N - 1$;



Definition of Regular Partition

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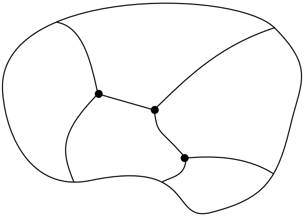
An open partition $(\omega_1, \dots, \omega_m)$ is called *regular* if:

1. denoting $\Gamma = \Omega \setminus \bigcup_{i=1}^m \omega_i$, there holds $\mathscr{H}_{\dim}(\Gamma) \leq N - 1$;
2. there exists a set $\mathcal{R} \subseteq \Gamma$, relatively open in Γ, such that
   - $\mathcal{R}$ is a collection of hypersurfaces of class $C^{1,\alpha}$, each one separating two different elements of the partition;
   - $\mathscr{H}_{\dim}(\Gamma \setminus \mathcal{R}) \leq N - 2$.



Outline

Main Result

Define the set of open partitions by

$$\mathcal{P}_m(\Omega) = \{(\omega_1, \dots, \omega_m) \subset \Omega^m : \ \omega_i \text{ open}, \ \omega_i \cap \omega_j = \emptyset \ i \neq j\}$$

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and consider the optimal partition problem

$$\inf_{(\omega_1,\dots,\omega_m)\in\mathcal{P}_m(\Omega)} F(\lambda_{k_1}(\omega_1), \dots, \lambda_{k_m}(\omega_m)). \qquad (1)$$

where the cost function: $F : (0, \infty)^m \to \mathbb{R}$ is of class C^1, and:

(F1) $\dfrac{\partial F}{\partial x_i} > 0$ in $(\mathbb{R}^+)^m$;

(F2) $F(x_1, \dots, x_{i-1}, x_i, x_{i+1} \dots, x_m) \to +\infty \quad \text{as} \quad x_i \to +\infty.$

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Theorem (Ramos, Tavares, T.)

The optimal partition problem (1) *admits a regular solution* $(\widetilde{\omega}_1, \dots, \widetilde{\omega}_m) \in \mathcal{P}_m(\Omega)$.

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Theorem (cont.)

Moreover, for each $i = 1, \dots, m$ *there exists* $1 \le l_i \le k_i$ *and*

- $\tilde{u}^i_1, \dots, \tilde{u}^i_{l_i}$ *eigenfunctions associated to the eigenvalue* $\lambda_{k_i}(\widetilde{\omega}_i)$*;*
- *coefficients* $\tilde{a}^i_1, \dots, \tilde{a}^i_{l_i} > 0$

such that

- $\tilde{u}^i_1, \dots, \tilde{u}^i_{l_i}$ *are* **Lipschitz continuous***;*
- $\widetilde{\omega}_i = int\left(\overline{\left\{\sum_{n=1}^{l_i} (\tilde{u}^i_n)^2 > 0\right\}}\right)$
- **Extremality condition on the regular part of the boundary (Weak Reflection Law)***:*

  given $x_0 \in \mathcal{R}$*, denoting by* $\widetilde{\omega}_i$ *and* $\widetilde{\omega}_j$ *the two adjacent sets of the partition at* x_0*,*

$$\lim_{\substack{x \to x_0 \\ x \in \widetilde{\omega}_i}} \sum_{n=1}^{l_i} \tilde{a}^i_n |\nabla \tilde{u}^i_n(x)|^2 = \lim_{\substack{x \to x_0 \\ x \in \widetilde{\omega}_j}} \sum_{n=1}^{l_j} \tilde{a}^j_n |\nabla \tilde{u}^j_n(x)|^2 \neq 0.$$

Example

2-partitions minimizing the sum of kth eigenvalues

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Fix $k \in \mathbb{N}$.

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} \left(\lambda_k(\omega_1) + \lambda_k(\omega_2)\right)$$

with

$$\mathcal{P}_2(\Omega) = \{(\omega_1,\omega_2) : \ \omega_1,\omega_2 \subset \Omega \text{ open}, \ \omega_1 \cap \omega_2 = \emptyset\}\,.$$

Outline

References

General results for any k

- [Bucur, Buttazzo, Henrot, Adv. Math. Sci. Appl. (1998)]
  - existence in the class of *quasi-open* sets
  - γ and weak γ–convergence, direct methods
- [Bourdin, Bucur, Oudet, SIAM J. Sci. Comp. (2009)]
  - existence in the class of *open* sets for $N = 2$
  - penalization with partition of the unity functions

References

The case of first eigenvalues

Sum of first eigenvalues: $k = 1$.

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} \left(\lambda_1(\omega_1) + \lambda_1(\omega_2)\right)$$

1st approach

- [Conti, T., Verzini, CVPDE (2005)]
- [Caffarelli, F.H. Lin, J. Sci. Comp. (2007)]
- [Tavares, T., CVPDE (2012)]

$$\inf \left\{ \int_\Omega (|\nabla u|^2 + |\nabla v|^2) : \; u, v \in H^1_0(\Omega), \; \int_\Omega u^2 = \int_\Omega v^2 = 1, \; u \cdot v \equiv 0 \right\}$$

Optimal partition problems related to the first eigenvalue

Next we consider some optimal partition problems involving the first eigenvalue. For any integer $m \geq 0$, we define the set of quasi-open m–partitions of Ω as

$$\mathcal{B}_m = \{(\omega_1, \ldots, \omega_m) : \ \omega_i \text{ quasi-open }, \ |\omega_i \cap \omega_j| = 0 \text{ for } i \neq j \text{ and } \cup_i \omega_i \subseteq \Omega\} .$$

Consider the following optimization problems: for any positive real number $p \geq 1$,

$$\mathcal{L}_{m,p} := \inf_{\mathcal{B}_m} \left(\frac{1}{h} \sum_{i=1}^{m} (\lambda_1(\omega_i))^p \right)^{1/p},$$

and, for $p = +\infty$ we find the limiting problem

$$\mathcal{L}_m := \inf_{\mathcal{B}_m} \max_{i=1,\ldots,m} (\lambda_1(\omega_i)),$$

where $\lambda_1(\omega)$ denotes the first eigenvalue of $-\Delta$ in $H_0^1(\omega)$ in a generalized sense.

Courant sharpness and deficiency

In some few cases, in order to compute $\mathfrak{L}_m(\Omega)$, one can look at the nodal partition associated with an eigenvalue.

Theorem (B. Helffer, T. Hoffmann-Ostenhof, S. T. Ann. IHP 2009)

If the graph of a minimal partition is bipartite, then it is the nodal domain of an eigenfunction φ_j.

Theorem (B. Helffer, T. Hoffmann-Ostenhof, S. T. (2009-10))

The m–th eigenfunction has exactly m nodal domains (i.e. is sharp with respect to the Courant nodal Theorem) if and only if the associated nodal m–partition is optimal with respect to the spectral m-th number.

G. Berkolaiko, P. Kuchment and U. Smilanski (2012) proved that *generically* the deficiency of nodal domains of the m-th eigenfunction is equal to the Morse index (in a suitable definition) of the associated partition, withe respect to the cost function of the minimal partition problem.

Extremality conditions

Our theorem applies to suitable multiples of the eigenfunctions associated with the optimal partition. More precisely, we proved that

Theorem (Conti, T., Verzini 2005, Helffer, Hoffmann-Ostenhof, T. 2009)

1. *Let $p \in [1, +\infty)$ and let $(\omega_1, \dots, \omega_m) \in \mathcal{B}_m$ be any minimal partition associated with $\mathcal{L}_{m,p}$ and let $(\phi_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\omega_i))_i$. Then there exist $a_i > 0$ such that the functions $u_i = a_i \phi_i$ verify in Ω, for every $i = 1, \dots, m$, the differential inequalities (in the distributional sense): $-\Delta u_i \leq \lambda_1(\omega_i) u_i$ and $-\Delta(u_i - \sum_{j \neq i} u_j) \geq \lambda_1(\omega_i) u_i - \sum_{j \neq i} \lambda_1(\omega_i) u_j$.*
2. *Let $(\tilde{\omega}_1, \dots, \tilde{\omega}_h) \in \mathcal{B}_m$ be any minimal partition associated with $\mathcal{L}_m$ and let $(\tilde{\phi}_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\tilde{\omega}_i))_i$. Then there exist $a_i \geq 0$, not all vanishing, such that the functions $\tilde{u}_i = a_i \tilde{\phi}_i$ verify in Ω, for every $i = 1, \dots, m$, the differential inequalities (in the distributional sense): $-\Delta \tilde{u}_i \leq \mathcal{L}_m \tilde{u}_i$ and $-\Delta(\tilde{u}_i - \sum_{j \neq i} \tilde{u}_j) \geq \mathcal{L}_m(\tilde{u}_i - \sum_{j \neq i} \tilde{u}_j)$.*

Regularity of the nodal set and the corresponding eigenfunctions

As consequence, we have the following result:

Theorem (Conti, T. Verzini 2005, Karakayan, Caffarelli, Lin 2008)

Let $(\omega_1, \ldots, \omega_h) \in \mathcal{B}_m$ be any minimal partition; then it admits an open, regular representative. The associate eigenfunctions are Lipschitz and the Weak Reflection Law holds.

References

Stronger results in special cases

2nd approach: eigenfunctions as limiting profiles of solutions to singularly perturbed systems of competition type

- [Chang, Lin, Lin Lin, Phys. D (2004)]
- [Conti, T., Verzini, CVPDE (2005)]
- [Tavares, T. AIHP (2012)]

$$\begin{cases} -\Delta u = \lambda_\beta u - \beta u v^2 \\ -\Delta v = \mu_\beta v - \beta u^2 v \\ u, v \in H_0^1(\Omega), \quad \displaystyle\int_\Omega u^2 = \int_\Omega v^2 = 1 \quad (\beta > 0) \end{cases}$$

Gradient System:

$$E_\beta(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2) + \frac{\beta}{2}\int_\Omega u^2 v^2$$

Phase Separation as $\beta \to +\infty$

The relation between both problems has been underlined in

- [Noris, Tavares, T., Verzini CPAM (2010)]
- [Tavares. T. CVPDE (2012)]

which imply (among other things) the following:

Theorem ($\beta \to +\infty$)

Let (u_β, v_β) be a minimal energy solution: $\inf_{\int u^2 = \int v^2 = 1} E_\beta(u, v)$.

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Let (u_β, v_β) be a minimal energy solution: $\inf_{\int u^2=\int v^2=1} E_\beta(u,v)$. Then there exist u, v, Lipschitz continuous, such that

- $u_\beta \to u$, $v_\beta \to v$ *in* $C^{0,\alpha} \cap H^1_0$;
- $u \cdot v \equiv 0$, *so* $(\{u > 0\}, \{v > 0\})$ *is an open partition;*
- $-\Delta u = \lambda u$ *in* $\{u > 0\}$, $-\Delta v = \lambda v$ *in* $\{v > 0\}$;
- $\Gamma := \{u = v = 0\}$ *is, up to a residual set, of class* $C^{1,\alpha}$.

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- $\Gamma := \{u=v=0\}$ *is, up to a residual set, of class* $C^{1,\alpha}$.

As $\beta \to +\infty$,

$$\inf_{\int u^2=\int v^2=1} E_\beta(u,v) \to \inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} \left(\lambda_1(\omega_1)+\lambda_1(\omega_2)\right)$$

How about higher eigenvalues? The case of the second eigenvalue is much simpler. It has been seen that:

$$\begin{cases} -\Delta u = \lambda_\beta u - \beta u v^2 \\ -\Delta v = \mu_\beta v - \beta u^2 v \\ u, v \in H_0^1(\Omega), \quad \int_\Omega u^2 = \int_\Omega v^2 = 1 \end{cases}$$

$$\Big\downarrow \; \beta \to +\infty$$

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} (\lambda_2(\omega_1) + \lambda_2(\omega_2)) \quad \text{or} \quad \inf_{(\omega_1,\omega_2)\in\mathcal{P}_2(\Omega)} (\lambda_1(\omega_1) + \lambda_2(\omega_2))$$

Reference:

- [Tavares, T. AIHP (2012)]
  - roughly speaking, one takes the least energy nodal solution of the system for each $\beta > 0$.

Extremality conditions for partitions involving higher eigenvalues

We would like to attack the optimal partition problem for higher eigenvalues ($k \geq 2$):

$$\mathcal{L} = \min \left(\sum_{i=1}^{m} \lambda_k(\omega_i) \right) .$$

Introduce the penalized functional:

$$E_\beta(u_1, \cdots, u_m) = \int_\Omega \sum_i |\nabla u_i|^2 + \beta \sum_{i \neq j} |u_i|^2 |u_j|^2$$

with constraints

$$\int_\Omega |u_i|^2 = 1 \quad \forall\, i = 1, \cdots, m .$$

As $\beta \to +\infty$, critical points of E_β converge to pairs of segregated eigenfunctions.

Main problems:

1. how to define a appropriate critical levels for the penalized functional?
2. how we derive coefficients for the Weak Reflection Law?

In the caso of partitions for the first eigenvalue, the Weak Reflection Law is a consequence of the domain variation formula.

Domain variations and the Weak Reflection Law

Assume U minimizes a Lagrangian energy with a pointwise constraint of the type $U(x) \in \Sigma$, for almost every $x \in \Omega$. Let $Y \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^N)$. Then, differentiation of the energy with respect to ε with $U(x) \mapsto U_\varepsilon(x) = U(x + \varepsilon Y(x))$ yields the well known identity ($\forall Y \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^N)$):

$$\int_\Omega \left\{ dY(x)\nabla U(x) \cdot \nabla U(x) - \operatorname{div} Y(x) \left[\frac{1}{2} |\nabla U(x)|^2 - F(U(x)) \right] \right\} = 0 \,,$$

By localizing to a regular bounded $\omega \subset \Omega$ this implies that, for every smooth ω and $\forall$ $Y \in \mathcal{C}^\infty(\Omega; \mathbb{R}^N)$

$$(*) \quad \int_\omega \left\{ dY(x)\nabla U(x) \cdot \nabla U(x) - \operatorname{div} Y(x) \left[\frac{1}{2} |\nabla U(x)|^2 - F(U(x)) \right] \right\} dx =$$
$$\int_{\partial\omega} \left\{ Y(x) \cdot \nabla U(x) \nu(x) \cdot \nabla U(x) - \nu(x) \cdot Y(x) \left[\frac{1}{2} |\nabla U(x)|^2 - F(U(x)) \right] \right\} d\sigma$$

Domain variations and the Weak Reflection Law

1. Identity $(*)$ yields the the Weak Reflection Law (whenever the nodal set is regular enough to integrate on)
2. Choose $Y(x) = x - x_0$ and $\omega = B_r(x_0)$:

$$(*) \quad + \quad \begin{pmatrix} Y(x) & = x - x_0 \\ \omega & = B_r(x_0) \end{pmatrix} \quad \Longrightarrow$$

Almgren's monotonity formula

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$$(*) \quad + \quad \begin{pmatrix} Y(x) & = x - x_0 \\ \omega & = B_r(x_0) \end{pmatrix} \quad \Longrightarrow \quad \text{Almgren's monotonity formula}$$

Two new problems:

- How to perform a domain variation for higher eigenvalues (in the case of degenerate eigenvalues).
- How to weight the eigenfunctions in the appropriate way.

Outline

Symmetric functions

Definition

We say that $\varphi \in \mathcal{F}$ if

1. $\varphi : \mathcal{S}_k(\mathbb{R}) \to \mathbb{R}$ is C^1 in $\mathcal{S}_k(\mathbb{R}) \setminus \{0\}$ and
$$\varphi(M) = \varphi(P^T MP) \qquad \text{for all } M \in \mathcal{S}_k(\mathbb{R}) \text{ and } P \in \mathcal{O}_k(\mathbb{R}).$$
2. Moreover, consider the restriction ψ of φ to the space of diagonal matrices, that is $\psi(a_1, \dots, a_k) := \varphi(\mathrm{diag}(a_1, \dots, a_k))$.

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2. Moreover, consider the restriction ψ of φ to the space of diagonal matrices, that is $\psi(a_1, \dots, a_k) := \varphi(\mathrm{diag}(a_1, \dots, a_k))$. Assume
   - $\frac{\partial \psi}{\partial a_i} > 0$ on $(\mathbb{R}^+)^k$ for every $i = 1, \dots, k$;
   - for each i and $\bar{a}_1, \dots, \bar{a}_{i-1}, \bar{a}_{i+1}, \dots, \bar{a}_k > 0$, we have

$$\psi(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \bar{a}_{i+1}, \dots, \bar{a}_k) \to +\infty \qquad \text{as } a_i \to +\infty.$$

Symmetric functions

Definition

We say that $\varphi \in \mathcal{F}$ if

1. $\varphi : \mathcal{S}_k(\mathbb{R}) \to \mathbb{R}$ is C^1 in $\mathcal{S}_k(\mathbb{R}) \setminus \{0\}$ and
$$\varphi(M) = \varphi(P^T MP) \qquad \text{for all } M \in \mathcal{S}_k(\mathbb{R}) \text{ and } P \in \mathcal{O}_k(\mathbb{R}).$$
2. Moreover, consider the restriction ψ of φ to the space of diagonal matrices, that is $\psi(a_1, \dots, a_k) := \varphi(\mathrm{diag}(a_1, \dots, a_k))$. Assume
   - $\frac{\partial \psi}{\partial a_i} > 0$ on $(\mathbb{R}^+)^k$ for every $i = 1, \dots, k$;
   - for each i and $\bar{a}_1, \dots, \bar{a}_{i-1}, \bar{a}_{i+1}, \dots, \bar{a}_k > 0$, we have
$$\psi(\bar{a}_1, \dots, \bar{a}_{i-1}, a_i, \bar{a}_{i+1}, \dots, \bar{a}_k) \to +\infty \quad \text{as } a_i \to +\infty.$$

Example:

$$\varphi(M) = (\mathrm{trace}(M^p))^{1/p} \Rightarrow \psi(a_1, \dots, a_m) = \left(\sum_{i=1}^{k} (a_i)^p\right)^{1/p}$$

A digression on the variational characterization of eigenvalues

Given $u \in H_0^1(\Omega;\mathbb{R}^k)$, define the $k \times k$ symmetric matrix

$$M(u) = \left(\int_\Omega \nabla u_i \cdot \nabla u_j \, dx \right)_{i,j=1,\dots,k}.$$

Our goal is to minimize

$$\min \left\{ \varphi(M(u)) \ : \ u = (u_1, \cdots, u_k) \in H_0^1(\Omega;\mathbb{R}^k) \, , \ \int_\Omega u_i u_j = \delta_{ij} \right\}$$

A trivial, but useful, remark is that:

Lemma

If $\varphi \in \mathcal{F}$, then the minimum is achieved in the class of u such that $M(u)$ is a diagonal matrix.

In other words, we have

$$\min\left\{\varphi\left(M(u)\right)\ :\ u=(u_1,\cdots,u_k)\ ,\ \int_\Omega u_iu_j=\delta_{ij}\right\}=$$

$$\min\left\{\varphi\left(M(u)\right)\ :\ u=(u_1,\cdots,u_k)\ ,\ \int_\Omega u_iu_j=\delta_{ij}\ \text{and}\ \int_\Omega \nabla u_i\cdot\nabla u_j=\delta_{ij}\right\}$$

Now, the $\mathcal{O}_k$-invariance of φ yields:

Lemma

If $\varphi(M)=\varphi(P^TMP)$ for every $M\in\mathcal{S}_k(\mathbb{R})$ and $P\in\mathcal{O}_k(\mathbb{R})$, then $\frac{\partial\varphi}{\partial\xi_{ij}}(D)=0$ for every diagonal matrix D, $j\neq i$.

Extremality conditions

Let $u = (u_1, \cdots, u_k) \in H^1_0(\Omega; \mathbb{R}^k)$ be a minimizer such that $M(u)$ is a diagonal matrix; then, there exist Lagrange multipliers (μ_{ij}) and $a_i > 0$ such that:

$$-a_i \Delta u_i = \sum_{j=1}^{k} \mu_{ij} u_j \ , \qquad \forall\, i = 1, \ldots, k$$

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with

$$a_i = \frac{\partial \varphi}{\partial \xi_{ii}}(M(u)) > 0.$$

One immediately sees that

$$M(u) \text{ diagonal} \Longrightarrow (\mu_{ij}) \text{ diagonal.}$$

Thus, denoting $\mu_i = \mu_{ii}$, we find:

$$-a_i \Delta u_i = \mu_i u_j \ , \qquad \forall\, i = 1, \ldots, k$$

Smooth and non symmetric functions

A subtlety is that not all smooth functions of the eigenvalues are smooth ($\mathcal{C}^1$) symmetric functions. Good examples are

$$\varphi(M) = \mathrm{trace}(M) = \sum_{i=1}^{k} \lambda_i \ , \qquad \varphi(M) = (\mathrm{trace}(M^p))^{1/p} = \left(\sum_{i=1}^{k} \lambda_i^p\right)^{1/p} .$$

But

$$\lambda_k = \max_{i=1,\dots,k} \lambda_i = \lim_{p\to+\infty} (\mathrm{trace}(M^p))^{1/p}$$

is only Lipschitz continuous. So, we have found a variational characterization of the k-th eigenvalue as a minimum (instead of minmax) of an energy at the expenses of regularity of the cost function. If the cost function is not smooth, we will approximate it with smooth one, and pass to the limit.

Back to the optimal partition problem: the $\varphi \in \mathcal{F}$ case

Let $\varphi \in \mathcal{F}$: Consider the penalized energy

$$E_\beta(u,v) = \varphi(M(u)) + \varphi(M(v)) + \frac{2\beta}{q}\int_\Omega (u_1^2 + \ldots + u_k^2)^{\frac{q}{2}} (v_1^2 + \ldots + v_k^2)^{\frac{q}{2}}\, dx$$

and consider the energy level

$$c_\beta = \inf\left\{ E_\beta(u,v) : \int_\Omega u_i u_j\, dx = \int_\Omega v_i v_j\, dx = \delta_{ij}\ \forall i,j \right\},$$

A general existence result

Lemma

Given u, v such that $\int_\Omega u_i u_j = \int_\Omega v_i v_j = \delta_{ij}$, there exist $\tilde{u}, \tilde{v}$ satisfying the same property and moreover:

- $\int_\Omega \nabla \tilde{u}_i \cdot \nabla \tilde{u}_j = \int_\Omega \nabla \tilde{v}_i \cdot \nabla \tilde{v}_j = 0 \quad \forall i \neq j$
- $\sum_{i=1}^k u_i^2 = \sum_{i=1}^k \tilde{u}_i^2$, $\sum_{i=1}^k v_i^2 = \sum_{i=1}^k \tilde{v}_i^2$ *pointwise.*
- *In particular,* $E_\beta(\tilde{u}, \tilde{v}) = E_\beta(u, v)$.

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- *In particular,* $E_\beta(\tilde u, \tilde v) = E_\beta(u, v)$.

Obs 1 $\tilde u = P^T u$, where P is the diagonalization matrix of $M(u) = (\int_\Omega \nabla u_i \cdot \nabla u_j)_{ij}$;

Obs 2 This justifies the shape of the competition term

$$\int_\Omega (u_1^2 + \ldots + u_k^2)^{\frac{q}{2}} (v_1^2 + \ldots + v_k^2)^{\frac{q}{2}}$$

A general existence result

Theorem (Existence of minimizers for each $\beta > 0$)

Given $\beta > 0$, the infimum c_β is attained at u_β, v_β such that

$$\int_\Omega \nabla u_{i,\beta} \cdot \nabla u_{j,\beta}\, dx = \int_\Omega \nabla v_{i,\beta} \cdot \nabla v_{j,\beta}\, dx = 0 \quad \text{whenever } i \neq j.$$

Moreover, for each i we have

$$\begin{cases} -a_{i,\beta}\Delta u_{i,\beta} = \sum_{j=1}^k \mu_{ij,\beta} u_{j,\beta} - \beta u_{i,\beta}\left(\sum_{j=1}^k u_{j,\beta}^2\right)^{\frac{q}{2}-1}\left(\sum_{j=1}^k v_{j,\beta}^2\right)^{\frac{q}{2}} \\ -b_{i,\beta}\Delta v_{i,\beta} = \sum_{j=1}^k \nu_{ij,\beta} v_{j,\beta} - \beta v_{i,\beta}\left(\sum_{j=1}^k v_{j,\beta}^2\right)^{\frac{q}{2}-1}\left(\sum_{j=1}^k u_{j,\beta}^2\right)^{\frac{q}{2}} \end{cases}$$

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with

$$a_{i,\beta} = \frac{\partial \varphi}{\partial \xi_{ii}}(M(u_\beta)), \qquad b_{i,\beta} = \frac{\partial \varphi}{\partial \xi_{ii}}(M(v_\beta)).$$

A general existence result

Theorem (Asymptotics as $\beta \to +\infty$)

There exists (u, v), Lipschitz continuous, such that, up to a subsequence, as $\beta \to +\infty$,

(i) $u_\beta \to u$, $v_\beta \to v$ *in* $C^{0,\alpha}(\overline{\Omega}) \cap H^1_0(\Omega)$;

(ii) $u_i \cdot v_j \equiv 0$ *in* Ω $\forall i, j$; $u, v \in \Sigma(L^2)$, *and*

$$\int_\Omega \beta \Big(\sum_{j=1}^k u_{j,\beta}^2\Big)^{\frac{q}{2}} \Big(\sum_{j=1}^k v_{j,\beta}^2\Big)^{\frac{q}{2}} \, dx \to 0.$$

(iii) *Moreover,*

$$-a_i \Delta u_i = \mu_i u_i \quad \text{in } \omega_u := \{x \in \Omega : u_1^2 + \ldots + u_k^2 > 0\},$$
$$-b_i \Delta v_i = \nu_i v_i \quad \text{in } \omega_v := \{x \in \Omega : v_1^2 + \ldots + v_k^2 > 0\}$$

for $a_i = \lim_\beta a_{i,\beta}$, $b_i = \lim_\beta b_{i,\beta}$, $\mu_i = \lim_\beta \mu_{ii,\beta}$, $\nu_i = \lim_\beta \nu_{ii,\beta}$.

Outline

A key tool: Almgren's monotonicity formula

Transversal to the proofs of existence and regularity. Recall that:

$$-a_i\Delta u_i = \mu_i u_i \quad \text{in } \omega_u := \{x \in \Omega : u_1^2 + \ldots + u_k^2 > 0\},$$
$$-b_i\Delta v_i = \nu_i v_i \quad \text{in } \omega_v := \{x \in \Omega : v_1^2 + \ldots + v_k^2 > 0\}$$

Define:

$$E(x_0,(u,v),r) = \frac{1}{r^{N-2}} \sum_{i=1}^{k} \int_{B_r(x_0)} \left(a_i|\nabla u_i|^2 + b_i|\nabla v_i|^2 - \mu_i u_i^2 - \nu_i v_i^2\right) dx$$

$$H(x_0,(u,v),r) = \frac{1}{r^{N-1}} \sum_{i=1}^{k} \int_{\partial B_r(x_0)} (a_i u_i^2 + b_i v_i^2)\, d\sigma$$

and the Almgren's quotient by

$$N(x_0,(u,v),r) = \frac{E(x_0,(u,v),r)}{H(x_0,(u,v),r)},$$

A key tool: Almgren's monotonicity formula

Theorem (Almgren's Monotonicity Formula)

Given $\tilde\Omega \Subset \Omega$, there exists $\tilde r > 0$ such that for every $x_0 \in \tilde\Omega$ and $r \in (0, \tilde r]$

$$\frac{d}{dr} N(x_0,(u,v),r) \geq -2Cr\left(N(x_0,(u,v),r)+1\right).$$

In particular,

- $e^{Cr^2}(N(x_0,(u,v),r)+1)$ *is a non decreasing function;*
- $N(x_0,(u,v),0^+) := \lim_{r\to 0^+} N(x_0,(u,v),r)$ *exists and is finite.*

Furthermore,

$$\frac{d}{dr}\log(H(x_0,(u,v),r)) = \frac{2}{r} N(x_0,(u,v),r) \qquad \forall r \in (0,\tilde r).$$

A key tool: Almgren's monotonicity formula
(and Local Pohožaev–type identities)

It is essential in several points:

- Liouville type theorems (a priori bounds);
- u, v are Lipschitz continuous;
- the nodal set $\Gamma_{(u,v)} = \{x \in \Omega : \ u_i(x) = v_i(x) = 0 \ \forall i\}$ (which corresponds to the common boundary of the sets of the partition) has empty interior
- convergence of blowup sequences, and characterization of its possible limits
- a priori characterization of the regular and singular parts of $\Gamma_{(u,v)}$.

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- a priori characterization of the regular and singular parts of $\Gamma_{(u,v)}$.

It is associated to the variational structure of the problem.

- Local Pohožaev–type identities

Regularity of the free boundary

General situation:

- u, v are Lipschitz continuous in $\overline{\Omega}$, $u_i \cdot v_j \equiv 0\ \forall i, j$;

Define

$$\Gamma_{(u,v)} := \{x \in \Omega : \ u_i(x) = v_i(x) = 0, \ \forall i = 1, \dots, k\} .$$

- In Ω,

$$-a_i \Delta u_i = \lambda_i u_i - \mathcal{M}_i \qquad - b_i \Delta v_i = \mu_i v_i - \mathcal{N}_i;$$

  with $\mathcal{M}_i$ and $\mathcal{N}_i$ are measures concentrated on $\Gamma_{(u,v)}$.
- Almgren's monotonicity formula (local Pohozaev-type identity)

Recall that the goal is:

Theorem

The nodal set $\Gamma_{(u,v)}$ splits in $\mathcal{R}_{(u,v)} \cup \mathcal{S}_{(u,v)}$, with

- *$\mathcal{R}_{(u,v)}$ is locally a $C^{1,\alpha}$–hypersurface*
- $\mathscr{H}_{dim}(\mathcal{S}_{(u,v)}) \le N - 2$

Regularity of the free boundary

Compactness of blowup sequences

Take some sequences $x_n \to x_0 \in \Omega$, $t_n \to 0^+$. We define a blowup sequence by

$$u_{i,n}(x) := \frac{u_i(x_n + t_n x)}{\rho_n}, \quad v_{i,n}(x) = \frac{v_i(x_n + t_n x)}{\rho_n} \quad \text{in } \Omega_n := \frac{\Omega - x_n}{t_n}$$

where we have normalized using the quantity

$$\rho_n^2 := H(x_n, (u,v), t_n) = \frac{1}{t_n^{N-1}} \sum_{i=1}^{k} \int_{\partial B_{t_n}(x_n)} (a_i u_i^2 + b_i v_i^2)\, d\sigma$$

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Theorem (convergence to a blowup limit)

$$(u_n, v_n) \to (\bar{u}, \bar{v}) \quad \text{in } C^{0,\alpha}_{loc}(\mathbb{R}^N) \cap H^1_{loc}(\mathbb{R}^N).$$

Regularity of the free boundary
Regular and Singular part

As u, v are Lipschitz continuous, one can check that

$$N(x,(u,v),0^+) \geq 1, \qquad \forall x \in \Gamma_{(u,v)}.$$

We use the Almgren's quotient to characterize a priori the regular and singular parts of the nodal set.

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Definition

We split the nodal set $\Gamma_{(u,v)}$ into the following two sets:

$$\mathcal{R}_{(u,v)} = \{x \in \Gamma_{(u,v)} : \ N(x,(u,v),0^+) = 1\}$$

and

$$\mathcal{S}_{(u,v)} = \{x \in \Gamma_{(u,v)} : \ N(x,(u,v),0^+) > 1\}.$$

Regular part of the free boundary

- If $N(x_0, (u, v), 0^+) = 1$, then one considers a blowup limit $(\bar{u}, \bar{v})$. There are two cases:

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  - if either $\bar u\equiv 0$ or $\bar v\equiv 0$, then we can prove that $u=0$ or $v=0$ for $x\sim x_0$ (Clean-Up)
  - both $\bar u,\bar v\not\equiv 0$, and $\Gamma_{(\bar u,\bar v)}$ is an hyperplane,

$$\bar u_i=\alpha_i(x\cdot\nu)^+,\qquad \bar v_i=\beta_i(x\cdot\nu)^-\qquad \text{in } \mathbb{R}^N.$$

Furthermore,

$$\sum_{i=1}^k a_i|\nabla\bar u_i|^2=\sum_{i=1}^k b_i|\nabla\bar v_i|^2 \text{ on the common boundary } \{x\cdot\nu=0\}$$

and so

$$\left(\sum_{i=1}^k a_i\bar u_i^2\right)^{1/2}-\left(\sum_{i=1}^k b_i\bar v_i^2\right)^{1/2}\qquad \text{is harmonic.}$$

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The question now is to know how to bring this to (u,v) at x_0.

Regularity of the free boundary

The set $\mathcal{R}$ is locally a regular hypersurface (following Caffarelli and Lin)

A replacement for the normal derivative:

The key is to study the vector:

$$\mathcal{U}(x) = \frac{U(x)}{|U(x)|} := \frac{(\sqrt{a_1}u_1(x), \ldots, \sqrt{a_k}u_k(x))}{\sqrt{a_1 u_1^2(x) + \ldots + a_k u_k^2(x)}}$$

- **Intuitively:** $\mathcal{U}(x_0) = \frac{\partial_\nu U(x_0)}{|\partial_\nu U(x_0)|}$ on the nodal set $\Gamma_{(u,v)}$

Rigorously, how can we define $\mathcal{U}(x_0)$ for $x_0 \in \Gamma_{(u,v)}$?

Actually, we can extend $\mathcal{U}(x)$ up to $\Gamma_{(u,v)}$ in a $C^{0,\alpha}$ way:

- Prove that (wlog) $u_1 > 0$ somewhere near each $x_0 \in \Gamma_{(u,v)}$;
- We can rewrite, for $x \notin \Gamma_{(u,v)}$,

$$\mathcal{U}(x) = \frac{\left(\sqrt{a_1}, \sqrt{a_2}\frac{u_2}{u_1}(x)\ldots, \sqrt{a_k}\frac{u_k}{u_1}(x)\right)}{\sqrt{a_1 + a_2\left(\frac{u_2}{u_1}(x)\right)^2 \ldots + a_k\left(\frac{u_k}{u_1}(x)\right)^2}}$$

- Prove a generalization of the **Boundary Harnack Principle** of [Jerison, Kenig, Adv. Math (1982)], showing that each $\frac{u_i}{u_1}$ is $C^{0,\alpha}$ up to the boundary

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  Now,

  - **More Rigorously:** $\mathcal{U}(x_0) = \frac{(\sqrt{a_1}\bar{u}_1, \dots, \sqrt{a_k}\bar{u}_k)}{|(\sqrt{a_1}\bar{u}_1, \dots, \sqrt{a_k}\bar{u}_k)|}$, where $\bar{u}$ is any blowup of u at x_0

Regularity of the free boundary

The set $\mathcal{R}$ is locally a regular hypersurface

Localize things at $x_0 \in \mathcal{R}$:

Definition

Given $x_0 \in \Gamma$ we define

$$u_{x_0}(x) = \mathcal{U}(x_0) \cdot U(x), \qquad v_{x_0}(x) = \mathcal{V}(x_0) \cdot V(x).$$

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When we zoom in at x_0, $u_{x_0}(x) - v_{x_0}(x)$ is close to an harmonic function whose nodal set is a hyperplane.

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When we zoom in at x_0, $u_{x_0}(x) - v_{x_0}(x)$ is close to an harmonic function whose nodal set is a hyperplane.

These new functions satisfy the following:

Lemma

There exist positive Radon measures $\mathcal{M}_{x_0}$, $\mathcal{N}_{x_0}$, both concentrated on Γ, such that

$$-\Delta u_{x_0} = \sum_{i=1}^{k} \frac{\mu_i}{a_i} \mathcal{U}_i(x_0) u_i - \mathcal{M}_{x_0}, \quad -\Delta v_{x_0} = \sum_{i=1}^{k} \frac{\nu_i}{b_i} \mathcal{V}_i(x_0) v_i - \mathcal{N}_{x_0}.$$

Regularity of the free boundary

The set $\mathcal{R}$ is locally a regular hypersurface

Let $\psi_{x_0,r}$, for each small $r > 0$, be the solution of

$$\begin{cases} -\Delta\psi_{x_0,r} = \sum_{i=1}^k \frac{\mu_i}{a_i}\mathcal{U}_i(x_0)u_i - \sum_{i=1}^k \frac{\nu_i}{b_i}\mathcal{V}_i(x_0)v_i & \text{in } B_r(x_0) \\ \psi_{x_0,r} = u_{x_0} - v_{x_0} & \text{on } \partial B_r(x_0). \end{cases}$$

Proposition

There exists

$$\nu(x_0) := \lim_{r\to 0} \nabla\psi_{x_0,r}(x_0).$$

Moreover, $\nu(x_0) \neq 0$ and the map $\Gamma \to \mathbb{R}^N$, $x_0 \mapsto \nu(x_0)$ is Hölder continuous of order α.

Regularity of the free boundary

The set $\mathcal{R}$ is locally a regular hypersurface

Theorem

The map

$$|U(x)| - |V(x)| = \sqrt{a_1 u_1^2 + \ldots + a_k u_k^2} - \sqrt{b_1 v_1^2 + \ldots + b_k v_k^2}$$

is differentiable at each $x_0 \in \mathcal{R}_{(u,v)}$*, with*

$$\nabla \left(|U| - |V|\right)(x_0) = \nu(x_0). \tag{3}$$

In particular, the set $\mathcal{R}_{(u,v)}$ *is locally a* $C^{1,\alpha}$*–hypersurface, for some* $\alpha \in (0,1)$.