Spectral geometry of the Steklov problem

Iosif Polterovich

Université de Montréal

Based on joint survey article with

Alexandre Girouard (Université Laval)

1. Steklov eigenvalue problem

1. Steklov eigenvalue problem

Let Ω be a Riemannian manifold.

1. Steklov eigenvalue problem

Let Ω be a Riemannian manifold. Consider the eigenvalue problem:

1. Steklov eigenvalue problem

Let Ω be a Riemannian manifold. Consider the eigenvalue problem:

$$\Delta u = 0 \quad \text{in } \Omega,$$

1. Steklov eigenvalue problem

Let Ω be a Riemannian manifold. Consider the eigenvalue problem:

$$\Delta u = 0 \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} = \sigma u \quad \text{on } M = \partial\Omega.$$

1. Steklov eigenvalue problem

Let Ω be a Riemannian manifold. Consider the eigenvalue problem:

$$\Delta u = 0 \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} = \sigma u \quad \text{on } M = \partial\Omega.$$

Steklov eigenvalues: $0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \nearrow +\infty$.

1. Steklov eigenvalue problem

Let Ω be a Riemannian manifold. Consider the eigenvalue problem:

$$\Delta u = 0 \text{ in } \Omega,$$

$$\frac{\partial u}{\partial n} = \sigma u \text{ on } M = \partial\Omega.$$

Steklov eigenvalues: $0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \nearrow +\infty$.

Eigenfunctions form an orthonormal basis in $L^2(M)$.

Motivation

Motivation

- σ_k are eigenvalues of the **Dirichlet–to–Neumann map** on M.

Motivation

- σ_k are eigenvalues of the **Dirichlet–to–Neumann map** on M.

The corresponding eigenfunctions are $u_k|_M$.

Motivation

- σ_k are eigenvalues of the **Dirichlet–to–Neumann map** on M.

The corresponding eigenfunctions are $u_k|_M$.

The DtN map is a first order elliptic ΨDO on M which has many applications.

Motivation

• σ_k are eigenvalues of the **Dirichlet–to–Neumann map** on M.

The corresponding eigenfunctions are $u_k|_M$.

The DtN map is a first order elliptic ΨDO on M which has many applications.

• Physical models: linear water waves, vibration of membranes, heat diffusion, etc.

Motivation

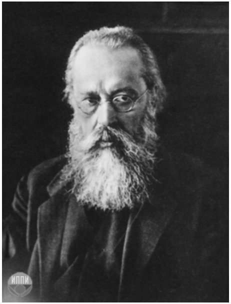
• σ_k are eigenvalues of the **Dirichlet–to–Neumann map** on M.

The corresponding eigenfunctions are $u_k|_M$.

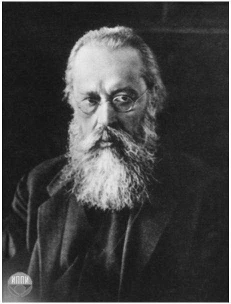
The DtN map is a first order elliptic ΨDO on M which has many applications.

• Physical models: linear water waves, vibration of membranes, heat diffusion, etc.

• Spectral geometry: Steklov eigenvalues and eigenfunctions have many distinctive features compared to their Dirichlet and Neumann counterparts.



Vladimir Steklov (1864-1926)



Vladimir Steklov (1864-1926)

V. Stekloff, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. Ecole Norm. Sup. **19** (1902), 455-490.

Examples

Examples

1. Steklov spectrum of a disk.

Eigenvalues: $0, 1, 1, 2, 2, 3, 3, \ldots, k, k, \ldots$

Examples

1. Steklov spectrum of a disk.

Eigenvalues: $0, 1, 1, 2, 2, 3, 3, \ldots, k, k, \ldots$

Eigenfunctions: $r^k \sin(k\phi)$, $r^k \cos(k\phi)$

Examples

1. Steklov spectrum of a disk.

Eigenvalues: $0, 1, 1, 2, 2, 3, 3, \ldots, k, k, \ldots$

Eigenfunctions: $r^k \sin(k\phi)$, $r^k \cos(k\phi)$

2. Steklov spectrum of a n–dimensional ball.

Eigenvalues: $0, 1, \ldots, 1, 2 \ldots, 2, \ldots, k, \ldots, k, \ldots$

Multiplicity of each eigenvalue is equal to the multiplicity of the eigenvalue $k(k+n-2)$ of the Laplacian on the boundary sphere.

Examples

1. Steklov spectrum of a disk.

Eigenvalues: $0, 1, 1, 2, 2, 3, 3, \ldots, k, k, \ldots$

Eigenfunctions: $r^k \sin(k\phi)$, $r^k \cos(k\phi)$

2. Steklov spectrum of a n–dimensional ball.

Eigenvalues: $0, 1, \ldots, 1, 2 \ldots, 2, \ldots, k, \ldots, k, \ldots$

Multiplicity of each eigenvalue is equal to the multiplicity of the eigenvalue $k(k + n - 2)$ of the Laplacian on the boundary sphere.

Eigenfunctions: $r^k u_k(\phi_1, \ldots, \phi_{n-1})$, u_k —corresponding spherical harmonic.

3. Steklov spectrum of a cylinder.

3. Steklov spectrum of a cylinder.

Let $\Omega = [-L, L] \times \Sigma \subset \mathbb{R} \times \Sigma$, where Σ is a closed Riemannian manifold.

3. Steklov spectrum of a cylinder.

Let $\Omega = [-L, L] \times \Sigma \subset \mathbb{R} \times \Sigma$, where Σ is a closed Riemannian manifold.

Let λ_k, k=1,2,... be the nonzero eigenvalues of the Laplacian on Σ with the corresponding eigenfunctions f_k.

3. Steklov spectrum of a cylinder.

Let $\Omega = [-L, L] \times \Sigma \subset \mathbb{R} \times \Sigma$, where Σ is a closed Riemannian manifold.

Let λ_k, k=1,2,... be the nonzero eigenvalues of the Laplacian on Σ with the corresponding eigenfunctions f_k.

Steklov eigenvalues of Ω:

$$0, \ 1/L, \ \sqrt{\lambda_k}\tanh(\sqrt{\lambda_k}L), \ \sqrt{\lambda_k}\coth(\sqrt{\lambda_k}L).$$

3. Steklov spectrum of a cylinder.

Let $\Omega = [-L, L] \times \Sigma \subset \mathbb{R} \times \Sigma$, where Σ is a closed Riemannian manifold.

Let λ_k, k=1,2,... be the nonzero eigenvalues of the Laplacian on Σ with the corresponding eigenfunctions f_k.

Steklov eigenvalues of Ω:

$$0,\ 1/L,\ \sqrt{\lambda_k}\tanh(\sqrt{\lambda_k}L),\ \sqrt{\lambda_k}\coth(\sqrt{\lambda_k}L).$$

Steklov eigenfunctions:

$$1, t, \cosh(\sqrt{\lambda_k}t)f_k(x),\ \sinh(\sqrt{\lambda_k}t)f_k(x).$$

4. Steklov spectrum of a disjoint union of disks.

4. Steklov spectrum of a disjoint union of disks.

Let Ω be the disjoint union of k disks of radii $r_1, \cdots, r_k$.
The spectrum of Ω is the disjoint union of their spectra.

4. Steklov spectrum of a disjoint union of disks.

Let Ω be the disjoint union of k disks of radii $r_1, \cdots, r_k$. The spectrum of Ω is the disjoint union of their spectra.

Let $G = \{\alpha_1, \cdots, \alpha_k\}$ where $\alpha_i = 1/r_i$.

4. Steklov spectrum of a disjoint union of disks.

Let Ω be the disjoint union of k disks of radii $r_1, \cdots, r_k$.
The spectrum of Ω is the disjoint union of their spectra.

Let $G = \{\alpha_1, \cdots, \alpha_k\}$ where $\alpha_i = 1/r_i$.

The spectrum of Ω can be represented as a **union of arithmetic progressions**

$$S(G) := \overbrace{\{0, \ldots\ldots, 0\}}^{k} \cup \alpha_1\mathbb{N} \cup \alpha_1\mathbb{N} \cup \alpha_2\mathbb{N} \cup \alpha_2\mathbb{N} \cup \cdots \cup \alpha_k\mathbb{N} \cup \alpha_k\mathbb{N},$$

reordered as a monotone increasing sequence.

2. Eigenvalue Asymptotics

2. Eigenvalue Asymptotics

Weyl's law (Hörmander): On any smooth manifold,

$$\sigma_k = 2\pi \left(\frac{k}{\mathrm{Vol}(M)\mathrm{Vol}(\mathbb{B}^{n-1})} \right)^{\frac{1}{n-1}} + O(1).$$

2. Eigenvalue Asymptotics

Weyl's law (Hörmander): On any smooth manifold,

$$\sigma_k = 2\pi\left(\frac{k}{\mathrm{Vol}(M)\mathrm{Vol}(\mathbb{B}^{n-1})}\right)^{\frac{1}{n-1}} + O(1).$$

Theorem (Rozenblium '79, Guillemin–Melrose '86) Let Ω be a smooth simply connected planar domain with perimeter L. Then

$$\sigma_{2k} = \frac{2\pi k}{L} + o(k^{-\infty}) = \sigma_{2k-1} + o(k^{-\infty}).$$

2. Eigenvalue Asymptotics

Weyl's law (Hörmander): On any smooth manifold,

$$\sigma_k = 2\pi \left(\frac{k}{\mathrm{Vol}(M)\mathrm{Vol}(\mathbb{B}^{n-1})} \right)^{\frac{1}{n-1}} + O(1).$$

Theorem (Rozenblium '79, Guillemin–Melrose '86) Let Ω be a smooth simply connected planar domain with perimeter L. Then

$$\sigma_{2k} = \frac{2\pi k}{L} + o(k^{-\infty}) = \sigma_{2k-1} + o(k^{-\infty}).$$

In other words, the spectrum coincides with the spectrum of a **disk** of perimeter L up to a $o(k^{-\infty})$ error.

Our next result shows that the spectrum of an arbitrary surface with n boundary components of lengths $L_1, \dots L_n$ coincides up to $o(k^{-\infty})$ with the spectrum of the **disjoint union of n disks** of the corresponding perimeters:

Theorem (Girouard–Parnovski–P.–Sher '14) Let $L_1, \ldots, L_n$ be the lengths of boundary components. Set $G = \left\{ \frac{2\pi}{L_1}, \cdots, \frac{2\pi}{L_n} \right\}$.

Theorem (Girouard–Parnovski–P.–Sher '14) Let $L_1, \dots, L_n$ be the lengths of boundary components. Set $G = \left\{ \frac{2\pi}{L_1}, \cdots, \frac{2\pi}{L_n} \right\}$.

Then

$$\sigma_k = S(G)_k + o(k^{-\infty}).$$

Theorem (Girouard–Parnovski–P.–Sher '14) Let $L_1, \ldots, L_n$ be the lengths of boundary components. Set $G = \left\{\frac{2\pi}{L_1}, \cdots, \frac{2\pi}{L_n}\right\}$.

Then

$$\sigma_k = S(G)_k + o(k^{-\infty}).$$

Recall that

$$S(G) := \overbrace{\{0, \ldots\ldots, 0\}}^{k} \cup \alpha_1\mathbb{N} \cup \alpha_1\mathbb{N} \cup \alpha_2\mathbb{N} \cup \alpha_2\mathbb{N} \cup \cdots \cup \alpha_k\mathbb{N} \cup \alpha_k\mathbb{N}.$$

listed in an increasing order.

Theorem (Girouard–Parnovski–P.–Sher '14) Let $L_1, \ldots, L_n$ be the lengths of boundary components. Set $G = \left\{\frac{2\pi}{L_1}, \cdots, \frac{2\pi}{L_n}\right\}$.

Then

$$\sigma_k = S(G)_k + o(k^{-\infty}).$$

Recall that

$$S(G) := \overbrace{\{0, \ldots \ldots, 0\}}^{k} \cup \alpha_1\mathbb{N} \cup \alpha_1\mathbb{N} \cup \alpha_2\mathbb{N} \cup \alpha_2\mathbb{N} \cup \cdots \cup \alpha_k\mathbb{N} \cup \alpha_k\mathbb{N}.$$

listed in an increasing order. For example,

$S(\{1, \pi\}) = 0, 0, 1, 1, 2, 2, 3, 3, \pi, \pi, 4, 4, 5, 5, 6, 6, 2\pi, 2\pi, 7, 7 \ldots$

One can hear the boundary components

One can hear the boundary components

Corollary (Girouard–Parnovski–P.–Sher '14) Let Ω be a smooth compact Riemannian surface with boundary. The Steklov spectrum uniquely determines the **number** n of boundary components of Ω, as well as the **lengths** $L_1, \dots L_n$ of each boundary component.

One can hear the boundary components

Corollary (Girouard–Parnovski–P.–Sher '14) Let Ω be a smooth compact Riemannian surface with boundary. The Steklov spectrum uniquely determines the **number** n of boundary components of Ω, as well as the **lengths** $L_1, \dots L_n$ of each boundary component.

Moreover, the length L_{max} of the largest boundary component is given by

$$L_{max} = \frac{2\pi}{\limsup_{k\to\infty}(\sigma_{k+1} - \sigma_k)}.$$

Higher dimensions: a counterexample

Higher dimensions: a counterexample

$$T_{1,1} := \mathbb{R}^2/\mathbb{Z}^2 \qquad T_{2,1} := \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$

$$T_{2,2} := \mathbb{R}^2/(2\mathbb{Z})^2 \qquad T_{\sqrt{2},\sqrt{2}} := \mathbb{R}^2/(\sqrt{2}\mathbb{Z})^2$$

Higher dimensions: a counterexample

$$T_{1,1} := \mathbb{R}^2/\mathbb{Z}^2 \qquad T_{2,1} := \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$
$$T_{2,2} := \mathbb{R}^2/(2\mathbb{Z})^2 \qquad T_{\sqrt{2},\sqrt{2}} := \mathbb{R}^2/(\sqrt{2}\mathbb{Z})^2$$

Doyle–Rossetti '11: *The surfaces*

$$\Sigma_1 = T_{1,1} \sqcup T_{1,1} \sqcup T_{2,2} \qquad \Sigma_2 = T_{2,1} \sqcup T_{2,1} \sqcup T_{\sqrt{2},\sqrt{2}}$$

are Laplace–isospectral.

Higher dimensions: a counterexample

$$T_{1,1} := \mathbb{R}^2/\mathbb{Z}^2 \qquad T_{2,1} := \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$
$$T_{2,2} := \mathbb{R}^2/(2\mathbb{Z})^2 \qquad T_{\sqrt{2},\sqrt{2}} := \mathbb{R}^2/(\sqrt{2}\mathbb{Z})^2$$

Doyle–Rossetti '11: *The surfaces*

$$\Sigma_1 = T_{1,1} \sqcup T_{1,1} \sqcup T_{2,2} \qquad \Sigma_2 = T_{2,1} \sqcup T_{2,1} \sqcup T_{\sqrt{2},\sqrt{2}}$$

are Laplace–isospectral.

For any $L > 0$, $\Omega_1 = [0, L] \times \Sigma_1$ and $\Omega_2 = [0, L] \times \Sigma_2$ are **Steklov isospectral**.

Higher dimensions: a counterexample

$$T_{1,1} := \mathbb{R}^2/\mathbb{Z}^2 \qquad T_{2,1} := \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$
$$T_{2,2} := \mathbb{R}^2/(2\mathbb{Z})^2 \qquad T_{\sqrt{2},\sqrt{2}} := \mathbb{R}^2/(\sqrt{2}\mathbb{Z})^2$$

Doyle–Rossetti '11: *The surfaces*

$$\Sigma_1 = T_{1,1} \sqcup T_{1,1} \sqcup T_{2,2} \qquad \Sigma_2 = T_{2,1} \sqcup T_{2,1} \sqcup T_{\sqrt{2},\sqrt{2}}$$

are Laplace–isospectral.

For any $L > 0$, $\Omega_1 = [0, L] \times \Sigma_1$ and $\Omega_2 = [0, L] \times \Sigma_2$ are **Steklov isospectral**.

- $\partial\Omega_1$ has 4 components of area 1 and 2 components of area 4.

Higher dimensions: a counterexample

$$T_{1,1} := \mathbb{R}^2/\mathbb{Z}^2 \qquad T_{2,1} := \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$
$$T_{2,2} := \mathbb{R}^2/(2\mathbb{Z})^2 \qquad T_{\sqrt{2},\sqrt{2}} := \mathbb{R}^2/(\sqrt{2}\mathbb{Z})^2$$

Doyle–Rossetti '11: *The surfaces*

$$\Sigma_1 = T_{1,1} \sqcup T_{1,1} \sqcup T_{2,2} \qquad \Sigma_2 = T_{2,1} \sqcup T_{2,1} \sqcup T_{\sqrt{2},\sqrt{2}}$$

are Laplace–isospectral.

For any $L > 0$, $\Omega_1 = [0, L] \times \Sigma_1$ and $\Omega_2 = [0, L] \times \Sigma_2$ are **Steklov isospectral**.

- $\partial\Omega_1$ has 4 components of area 1 and 2 components of area 4.
- $\partial\Omega_2$ has 6 components of area 2.

Higher dimensions: a counterexample

$$T_{1,1} := \mathbb{R}^2/\mathbb{Z}^2 \qquad T_{2,1} := \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$
$$T_{2,2} := \mathbb{R}^2/(2\mathbb{Z})^2 \qquad T_{\sqrt{2},\sqrt{2}} := \mathbb{R}^2/(\sqrt{2}\mathbb{Z})^2$$

Doyle–Rossetti '11: *The surfaces*

$$\Sigma_1 = T_{1,1} \sqcup T_{1,1} \sqcup T_{2,2} \qquad \Sigma_2 = T_{2,1} \sqcup T_{2,1} \sqcup T_{\sqrt{2},\sqrt{2}}$$

are Laplace–isospectral.

For any $L > 0$, $\Omega_1 = [0, L] \times \Sigma_1$ and $\Omega_2 = [0, L] \times \Sigma_2$ are **Steklov isospectral**.

- $\partial\Omega_1$ has 4 components of area 1 and 2 components of area 4.
- $\partial\Omega_2$ has 6 components of area 2.

The collection of areas of boundary components is not a Steklov spectral invariant.

Higher dimensions: a counterexample

$$T_{1,1} := \mathbb{R}^2/\mathbb{Z}^2 \qquad T_{2,1} := \mathbb{R}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$

$$T_{2,2} := \mathbb{R}^2/(2\mathbb{Z})^2 \qquad T_{\sqrt{2},\sqrt{2}} := \mathbb{R}^2/(\sqrt{2}\mathbb{Z})^2$$

Doyle–Rossetti '11: *The surfaces*

$$\Sigma_1 = T_{1,1} \sqcup T_{1,1} \sqcup T_{2,2} \qquad \Sigma_2 = T_{2,1} \sqcup T_{2,1} \sqcup T_{\sqrt{2},\sqrt{2}}$$

are Laplace–isospectral.

For any $L > 0$, $\Omega_1 = [0, L] \times \Sigma_1$ and $\Omega_2 = [0, L] \times \Sigma_2$ are **Steklov isospectral**.

- $\partial\Omega_1$ has 4 components of area 1 and 2 components of area 4.
- $\partial\Omega_2$ has 6 components of area 2.

Open Problem: is the number of boundary components Steklov spectral invariant in dimensions $n \geq 3$?

3. Isoperimetric inequalities for Steklov eigenvalues

3. Isoperimetric inequalities for Steklov eigenvalues

Variational characterization of Steklov eigenvalues:

3. Isoperimetric inequalities for Steklov eigenvalues

Variational characterization of Steklov eigenvalues:

$$\sigma_k(\Omega) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_{\partial\Omega} u^2 \, ds}, \quad k = 1, 2, \ldots$$

3. Isoperimetric inequalities for Steklov eigenvalues

Variational characterization of Steklov eigenvalues:

$$\sigma_k(\Omega) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_{\partial\Omega} u^2 \, ds}, \quad k = 1, 2, \ldots$$

The infimum runs over all k-dimensional subspaces E_k of the Sobolev space $H^1(\Omega)$ which are orthogonal to constants on $\partial\Omega$.

3. Isoperimetric inequalities for Steklov eigenvalues

Variational characterization of Steklov eigenvalues:

$$\sigma_k(\Omega) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_{\partial\Omega} u^2 \, ds}, \quad k = 1, 2, \ldots$$

The infimum runs over all k-dimensional subspaces E_k of the Sobolev space $H^1(\Omega)$ which are orthogonal to constants on $\partial\Omega$.

An interesting problem is to **maximize** σ_k in the class of domains of the fixed volume of the boundary.

Weinstock's inequality

Theorem (Weinstock '54) Among all *simply connected* planar domains of given **perimeter**, the first nonzero Steklov eigenvalue attains its maximum iff the domain is a disk.

Weinstock's inequality

Theorem (Weinstock '54) Among all *simply connected* planar domains of given **perimeter**, the first nonzero Steklov eigenvalue attains its maximum iff the domain is a disk.

This result can be viewed as the analogue of the Szegő inequality for the first Neumann eigenvalue.

Weinstock's inequality

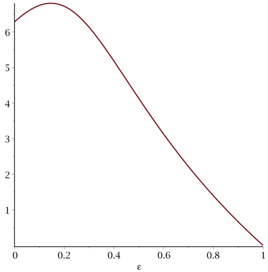
Theorem (Weinstock '54) Among all *simply connected* planar domains of given **perimeter**, the first nonzero Steklov eigenvalue attains its maximum iff the domain is a disk.

This result can be viewed as the analogue of the Szegő inequality for the first Neumann eigenvalue.

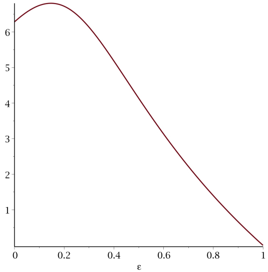
However, unlike Szegő's inequality, Weinstock's inequality does not extend to non-simply connected domains.

Example Normalized first eigenvalue of the annuli $A(1, \epsilon)$. The maximum is achieved for $\epsilon > 0$.

Example Normalized first eigenvalue of the annuli $A(1, \epsilon)$. The maximum is achieved for $\epsilon > 0$.



Example Normalized first eigenvalue of the annuli $A(1, \epsilon)$. The maximum is achieved for $\epsilon > 0$.



Open problem: What is the maximal value of the normalized first Steklov eigenvalue among all planar domains of fixed perimeter?

Inequalities for surfaces

Inequalities for surfaces

Theorem (Girouard -P. '13) Let Ω be compact surface of genus γ and with l boundary components. Then

$$\sigma_k(\Omega)L(\partial\Omega) \leq 2\pi(\gamma + l)k, \quad k = 1, 2, \ldots$$

Inequalities for surfaces

Theorem (Girouard -P. '13) Let Ω be compact surface of genus γ and with l boundary components. Then

$$\sigma_k(\Omega)L(\partial\Omega) \leq 2\pi(\gamma + l)k, \quad k = 1, 2, \dots$$

$\gamma = 0, l = 1, k = 1$: Weinstock inequality.

Inequalities for surfaces

Theorem (Girouard -P. '13) Let Ω be compact surface of genus γ and with l boundary components. Then

$$\sigma_k(\Omega)L(\partial\Omega) \leq 2\pi(\gamma + l)k, \quad k = 1, 2, \dots$$

$\gamma = 0, l = 1, k = 1$: Weinstock inequality.

$\gamma = 0, l = 1$: Hersch–Payne–Schiffer inequality (1975).

Inequalities for surfaces

Theorem (Girouard -P. '13) Let Ω be compact surface of genus γ and with l boundary components. Then

$$\sigma_k(\Omega)L(\partial\Omega) \leq 2\pi(\gamma + l)k, \quad k = 1, 2, \dots$$

$\gamma = 0, l = 1, k = 1$: Weinstock inequality.

$\gamma = 0, l = 1$: Hersch–Payne–Schiffer inequality (1975).
The equality is attained in the limit by a disjoint union of k identical disks (Girouard-P. '10).

Inequalities for surfaces

Theorem (Girouard -P. '13) Let Ω be compact surface of genus γ and with l boundary components. Then

$$\sigma_k(\Omega)L(\partial\Omega) \leq 2\pi(\gamma + l)k, \quad k = 1, 2, \ldots$$

$\gamma = 0, l = 1, k = 1$: Weinstock inequality.

$\gamma = 0, l = 1$: Hersch–Payne–Schiffer inequality (1975).
The equality is attained in the limit by a disjoint union of k identical disks (Girouard-P. '10).

$k = 1$: Fraser-Schoen '12.

Extremal metrics for Stekov eigenvalues

Extremal metrics for Stekov eigenvalues

Let $\sigma^\star(\gamma, l)$ be the supremum of $\sigma_1 L$ taken over all Riemannian metrics on a compact surface of genus γ with l boundary components.

Extremal metrics for Stekov eigenvalues

Let $\sigma^\star(\gamma, l)$ be the supremum of $\sigma_1 L$ taken over all Riemannian metrics on a compact surface of genus γ with l boundary components.

Theorem (Fraser-Schoen, '13) Let Ω be a compact surface with boundary and let g be a smooth metric on Ω such that

$$\sigma_1(\Omega, g)L(\partial\Omega, g) = \sigma^\star(\gamma, l).$$

Then there exist first eigenfunctions $u_1, \cdots, u_n$ such that the map

$$u = (u_1, \cdots, u_n) : \Omega \to \mathbb{B}^n$$

is a conformal minimal immersion into the ball $\mathbb{B}^n$ which is up to a rescaling an isometry on $\partial\Omega$, and such that $u(\Omega) \subset \mathbb{B}^n$ is a free boundary solution, i.e. $u(\Omega)$ is orthogonal to the boundary sphere.

Surfaces of genus zero

Surfaces of genus zero

Theorem (Fraser-Schoen '13) For each $l > 0$, there exists a smooth metric g on the surface of genus 0 with l boundary components such that

$$\sigma_1(\Omega, g) L(\partial\Omega, g) = \sigma^\star(0, l).$$

Surfaces of genus zero

Theorem (Fraser-Schoen '13) For each $l > 0$, there exists a smooth metric g on the surface of genus 0 with l boundary components such that

$$\sigma_1(\Omega, g) L(\partial\Omega, g) = \sigma^\star(0, l).$$

Moreover, the sequence $\sigma^\star(0, l)$ is monotone increasing and converges to 4π as $l \to \infty$.

Surfaces of genus zero

Theorem (Fraser-Schoen '13) For each $l > 0$, there exists a smooth metric g on the surface of genus 0 with l boundary components such that

$$\sigma_1(\Omega, g) L(\partial\Omega, g) = \sigma^\star(0, l).$$

Moreover, the sequence $\sigma^\star(0, l)$ is monotone increasing and converges to 4π as $l \to \infty$.The limiting surface is a double disk.

Surfaces of genus zero

Theorem (Fraser-Schoen '13) For each $l > 0$, there exists a smooth metric g on the surface of genus 0 with l boundary components such that

$$\sigma_1(\Omega, g)L(\partial\Omega, g) = \sigma^\star(0, l).$$

Moreover, the sequence $\sigma^\star(0, l)$ is monotone increasing and converges to 4π as $l \to \infty$.The limiting surface is a double disk.

Example On a topological annulus ($\gamma = 0, l = 2$), the maximal value of $\sigma_1(\Omega)L(\partial\Omega)$ is attained by the critical catenoid, which is the minimal surface $\Omega \subset B^3$ parametrized by

$$\phi(t, \theta) = c(\cosh(t)\cos(\theta), \cosh(t)\sin(\theta), t),$$

where the scaling factor $c > 0$ is chosen so that the boundary of the surface Ω meets the sphere $\mathbb{S}^2$ orthogonally.

Higher dimensions

Higher dimensions

In dimensions $n \geq 3$ geometric bounds on Steklov eigenvalues are more complicated and involve other geometric quantities, such as the isoperimetric ratio.

Higher dimensions

In dimensions $n \geq 3$ geometric bounds on Steklov eigenvalues are more complicated and involve other geometric quantities, such as the isoperimetric ratio.

Open problem: Does there exist a constant C_n such that, for any n-dimensional Euclidean domain,

$$\sigma_k(\Omega)|\partial\Omega|^{1/n} \leq C_n k^{\frac{1}{n-1}}?$$

Higher dimensions

In dimensions $n \geq 3$ geometric bounds on Steklov eigenvalues are more complicated and involve other geometric quantities, such as the isoperimetric ratio.

Open problem: Does there exist a constant C_n such that, for any n-dimensional Euclidean domain,

$$\sigma_k(\Omega)|\partial\Omega|^{1/n} \leq C_n k^{\frac{1}{n-1}}?$$

This question was posed by Colbois–El Soufi–Girouard '11.

Higher dimensions

In dimensions $n \geq 3$ geometric bounds on Steklov eigenvalues are more complicated and involve other geometric quantities, such as the isoperimetric ratio.

Open problem: Does there exist a constant C_n such that, for any n-dimensional Euclidean domain,

$$\sigma_k(\Omega)|\partial\Omega|^{1/n} \leq C_n k^{\frac{1}{n-1}}?$$

This question was posed by Colbois–El Soufi–Girouard '11.

The same authors proved a similar bound with the exponent $2/n$.

Higher dimensions

In dimensions $n \geq 3$ geometric bounds on Steklov eigenvalues are more complicated and involve other geometric quantities, such as the isoperimetric ratio.

Open problem: Does there exist a constant C_n such that, for any n-dimensional Euclidean domain,

$$\sigma_k(\Omega)|\partial\Omega|^{1/n} \leq C_n k^{\frac{1}{n-1}}?$$

This question was posed by Colbois–El Soufi–Girouard '11.

The same authors proved a similar bound with the exponent $2/n$.

The problem is to find a bound consistent with *Weyl's law*.

4. Can one hear the shape of a Steklov drum?

4. Can one hear the shape of a Steklov drum?

As we have seen earlier, in general the answer is NO. Recall:

4. Can one hear the shape of a Steklov drum?

As we have seen earlier, in general the answer is NO. Recall:

Construction 1. Let Σ_1 and Σ_2 be two isospectral closed Riemannian manifolds.

4. Can one hear the shape of a Steklov drum?

As we have seen earlier, in general the answer is NO. Recall:

Construction 1. Let Σ_1 and Σ_2 be two isospectral closed Riemannian manifolds.
Then $\Omega_1 = [-L, L] \times \Sigma_1$ and $\Omega_2 = [-L, L] \times \Sigma_2$ are Steklov isospectral.

4. Can one hear the shape of a Steklov drum?

As we have seen earlier, in general the answer is NO. Recall:

Construction 1. Let Σ_1 and Σ_2 be two isospectral closed Riemannian manifolds.
Then $\Omega_1 = [-L, L] \times \Sigma_1$ and $\Omega_2 = [-L, L] \times \Sigma_2$ are Steklov isospectral.

Construction 2. Let Ω be a surface with boundary, and let g_1 and g_2 be conformally equivalent Riemannian metrics on Ω which coincide on $\partial\Omega$.

4. Can one hear the shape of a Steklov drum?

As we have seen earlier, in general the answer is NO. Recall:

Construction 1. Let Σ_1 and Σ_2 be two isospectral closed Riemannian manifolds.
Then $\Omega_1 = [-L, L] \times \Sigma_1$ and $\Omega_2 = [-L, L] \times \Sigma_2$ are Steklov isospectral.

Construction 2. Let Ω be a surface with boundary, and let g_1 and g_2 be conformally equivalent Riemannian metrics on Ω which coincide on $\partial\Omega$. Then (Ω, g_1) and (Ω, g_2) are Steklov isospectral.

4. Can one hear the shape of a Steklov drum?

As we have seen earlier, in general the answer is NO. Recall:

Construction 1. Let Σ_1 and Σ_2 be two isospectral closed Riemannian manifolds.
Then $\Omega_1 = [-L, L] \times \Sigma_1$ and $\Omega_2 = [-L, L] \times \Sigma_2$ are Steklov isospectral.

Construction 2. Let Ω be a surface with boundary, and let g_1 and g_2 be conformally equivalent Riemannian metrics on Ω which coincide on $\partial\Omega$. Then (Ω, g_1) and (Ω, g_2) are Steklov isospectral.

Follows from conformal invariance of the Dirichlet energy and variational charaterization of Steklov eigenvalues.

4. Can one hear the shape of a Steklov drum?

As we have seen earlier, in general the answer is NO. Recall:

Construction 1. Let Σ_1 and Σ_2 be two isospectral closed Riemannian manifolds.
Then $\Omega_1 = [-L, L] \times \Sigma_1$ and $\Omega_2 = [-L, L] \times \Sigma_2$ are Steklov isospectral.

Construction 2. Let Ω be a surface with boundary, and let g_1 and g_2 be conformally equivalent Riemannian metrics on Ω which coincide on $\partial\Omega$. Then (Ω, g_1) and (Ω, g_2) are Steklov isospectral.

Follows from conformal invariance of the Dirichlet energy and variational charaterization of Steklov eigenvalues.

More examples: Gordon–Herbrich–Webb '15 have recently adapted Sunada's method to the Steklov setting.

Open Problem: Do there exist Steklov isospectral planar domains?

Open Problem: Do there exist Steklov isospectral planar domains?

Proposition If a smooth planar domain is Steklov isospectral to a disk, it is a disk of the same radius.

Open Problem: Do there exist Steklov isospectral planar domains?

Proposition If a smooth planar domain is Steklov isospectral to a disk, it is a disk of the same radius.

Proof. Indeed, it is simply-connected, since the number of boundary components is a Steklov spectral invariant.

Open Problem: Do there exist Steklov isospectral planar domains?

Proposition If a smooth planar domain is Steklov isospectral to a disk, it is a disk of the same radius.

Proof. Indeed, it is simply-connected, since the number of boundary components is a Steklov spectral invariant. It has the same perimeter by Weyl's law.

Open Problem: Do there exist Steklov isospectral planar domains?

Proposition If a smooth planar domain is Steklov isospectral to a disk, it is a disk of the same radius.

Proof. Indeed, it is simply-connected, since the number of boundary components is a Steklov spectral invariant. It has the same perimeter by Weyl's law. By Weinstock's inequality, the first eigenvalue is maximized iff the domain is a disk. □

Conjecture (P. -Sher '14) A Euclidean ball is uniquely determined by its Steklov spectrum in any dimension.

Conjecture (P. -Sher '14) A Euclidean ball is uniquely determined by its Steklov spectrum in any dimension.

Proved in dimension 3 for smooth domains with connected boundaries.

Conjecture (P. -Sher '14) A Euclidean ball is uniquely determined by its Steklov spectrum in any dimension.

Proved in dimension 3 for smooth domains with connected boundaries. The proof uses explict expressions for the Steklov heat invariants

Conjecture (P. -Sher '14) A Euclidean ball is uniquely determined by its Steklov spectrum in any dimension.

Proved in dimension 3 for smooth domains with connected boundaries. The proof uses explict expressions for the Steklov heat invariants and a generalization of a theorem due to Zelditch that maximal multiplicity growth implies periodicity of the geodesic flow.

Conjecture (P. -Sher '14) A Euclidean ball is uniquely determined by its Steklov spectrum in any dimension.

Proved in dimension 3 for smooth domains with connected boundaries. The proof uses explict expressions for the Steklov heat invariants and a generalization of a theorem due to Zelditch that maximal multiplicity growth implies periodicity of the geodesic flow.

Remark In the Laplace-Beltrami setting, a similar conjecture is open for spheres in dimensions $n \geq 7$.

Conjecture (P. -Sher '14) A Euclidean ball is uniquely determined by its Steklov spectrum in any dimension.

Proved in dimension 3 for smooth domains with connected boundaries. The proof uses explict expressions for the Steklov heat invariants and a generalization of a theorem due to Zelditch that maximal multiplicity growth implies periodicity of the geodesic flow.

Remark In the Laplace-Beltrami setting, a similar conjecture is open for spheres in dimensions $n \geq 7$.

Open problem: Does the Steklov spectrum of a manifold determine the Laplace spectrum of its boundary?

Conjecture (P. -Sher '14) A Euclidean ball is uniquely determined by its Steklov spectrum in any dimension.

Proved in dimension 3 for smooth domains with connected boundaries. The proof uses explicit expressions for the Steklov heat invariants and a generalization of a theorem due to Zelditch that maximal multiplicity growth implies periodicity of the geodesic flow.

Remark In the Laplace-Beltrami setting, a similar conjecture is open for spheres in dimensions $n \geq 7$.

Open problem: Does the Steklov spectrum of a manifold determine the Laplace spectrum of its boundary?

The answer is YES in dimension two.

5. Nodal geometry of Steklov eigenfunctions

5. Nodal geometry of Steklov eigenfunctions

Let, as before, Ω be an n-dimensional Riemannian manifold with boundary M.

5. Nodal geometry of Steklov eigenfunctions

Let, as before, Ω be an n-dimensional Riemannian manifold with boundary M.

Let ϕ_σ be an eigenfunction of the Steklov problem on Ω with eigenvalue σ.

5. Nodal geometry of Steklov eigenfunctions

Let, as before, Ω be an n-dimensional Riemannian manifold with boundary M.

Let ϕ_σ be an eigenfunction of the Steklov problem on Ω with eigenvalue σ.

Let $u_\sigma = \phi_\sigma|_M$ be the corresponding eigenfunction of the Dirichlet-to-Neumann operator on M.

5. Nodal geometry of Steklov eigenfunctions

Let, as before, Ω be an n-dimensional Riemannian manifold with boundary M.

Let ϕ_σ be an eigenfunction of the Steklov problem on Ω with eigenvalue σ.

Let $u_\sigma = \phi_\sigma|_M$ be the corresponding eigenfunction of the Dirichlet-to-Neumann operator on M.

Courant's theorem: The number of (interior) nodal domains of the n-th Steklov eigenfunction ϕ_{σ_n} is at most $n+1$.

Open problem: Find an upper bound for the number of nodal domains of the n-th eigenfunction u_{σ_n} of the Dirichlet-to-Neumann operator (boundary nodal domains).

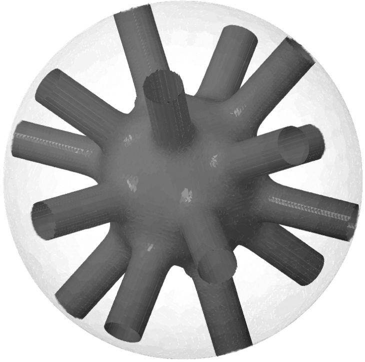
Open problem: Find an upper bound for the number of nodal domains of the n-th eigenfunction u_{σ_n} of the Dirichlet-to-Neumann operator (boundary nodal domains).

Remarks (i) The proof of Courant's theorem fails for the DtN operator, as it is nonlocal.

Open problem: Find an upper bound for the number of nodal domains of the n-th eigenfunction u_{σ_n} of the Dirichlet-to-Neumann operator (boundary nodal domains).

Remarks (i) The proof of Courant's theorem fails for the DtN operator, as it is nonlocal.

(ii) In two dimensions, the number of interior nodal domains controls the number of boundary nodal domains. However, in higher dimensions this is not the case.



Example A surface inside a ball creating two interior connected components and a large number of boundary connected components.

Density of the nodal sets

Density of the nodal sets

Open Problem: (i) Are the nodal sets of Steklov eigenfunctions on a Riemannian manifold Ω dense on the scale $1/\sigma$ in Ω?

Density of the nodal sets

Open Problem: (i) Are the nodal sets of Steklov eigenfunctions on a Riemannian manifold Ω dense on the scale $1/\sigma$ in Ω?

(ii) Are the nodal sets of the Dirichlet-to-Neumann eigenfunctions dense on the scale of $1/\sigma$ on $\partial\Omega$?

Density of the nodal sets

Open Problem: (i) Are the nodal sets of Steklov eigenfunctions on a Riemannian manifold Ω dense on the scale $1/\sigma$ in Ω?

(ii) Are the nodal sets of the Dirichlet-to-Neumann eigenfunctions dense on the scale of $1/\sigma$ on $\partial\Omega$?

Remark For smooth simply connected planar domains, the nodal sets are known to be dense on the boundary (Shamma '71).

Density of the nodal sets

Open Problem: (i) Are the nodal sets of Steklov eigenfunctions on a Riemannian manifold Ω dense on the scale $1/\sigma$ in Ω?

(ii) Are the nodal sets of the Dirichlet-to-Neumann eigenfunctions dense on the scale of $1/\sigma$ on $\partial\Omega$?

Remark For smooth simply connected planar domains, the nodal sets are known to be dense on the boundary (Shamma '71).

The question about the density of the interior nodal set is open even in this setting!

Size of the nodal sets

Size of the nodal sets

Open problem (interior and boundary "Yau's conjectures"):

Show that

Open problem (interior and boundary "Yau's conjectures"):
Show that

$$(i)\;\; C_2\sigma \leq \mathcal{H}_{n-1}(\mathcal{N}(\phi_\sigma)) \leq C_1\sigma,$$

Size of the nodal sets

Open problem (interior and boundary "Yau's conjectures"):
Show that

$$(i)\ \ C_2\sigma \le \mathcal{H}_{n-1}(\mathcal{N}(\phi_\sigma)) \le C_1\sigma,$$
$$(ii)\ \ C_2'\sigma \le \mathcal{H}_{n-2}(\mathcal{N}(u_\sigma)) \le C_1'\sigma,$$

Size of the nodal sets

Open problem (interior and boundary "Yau's conjectures"):
Show that

$$(i)\ \ C_2\sigma \leq \mathcal{H}_{n-1}(\mathcal{N}(\phi_\sigma)) \leq C_1\sigma,$$

$$(ii)\ \ C_2'\sigma \leq \mathcal{H}_{n-2}(\mathcal{N}(u_\sigma)) \leq C_1'\sigma,$$

where the constants C_1, C_2, C_1', C_2' depend only on the geometry of the manifold.

Size of the nodal sets

Open problem (interior and boundary "Yau's conjectures"):
Show that

$$(i) \;\; C_2\sigma \le \mathcal{H}_{n-1}(\mathcal{N}(\phi_\sigma)) \le C_1\sigma,$$
$$(ii) \;\; C_2'\sigma \le \mathcal{H}_{n-2}(\mathcal{N}(u_\sigma)) \le C_1'\sigma,$$

where the constants C_1, C_2, C_1', C_2' depend only on the geometry of the manifold.

The upper bound in (ii) was conjectured by Bellova–Lin '14 and proved by Zelditch '14 for real analytic manifolds. Some non-sharp lower bounds in (ii) were recently proved by Wang–Zhu '14, and in (i) by Sogge–Wang–Zhu '15.

Nodal length of Steklov eigenfunctions on real analytic surfaces

Nodal length of Steklov eigenfunctions on real analytic surfaces

Part (i) of the open problem has been recently solved for real analytic surfaces:

Theorem (P.–Sher–Toth '15) Let M be a real-analytic Riemannian surface. Then there exist positive constants C_1, C_2 such that:

Nodal length of Steklov eigenfunctions on real analytic surfaces

Part (i) of the open problem has been recently solved for real analytic surfaces:

Theorem (P.–Sher–Toth '15) Let M be a real-analytic Riemannian surface. Then there exist positive constants C_1, C_2 such that:

$$C_2\sigma \leq \mathrm{Length}(\mathcal{N}(\phi_\sigma)) \leq C_1\sigma.$$

Nodal length of Steklov eigenfunctions on real analytic surfaces

Part (i) of the open problem has been recently solved for real analytic surfaces:

Theorem (P.–Sher–Toth '15) Let M be a real-analytic Riemannian surface. Then there exist positive constants C_1, C_2 such that:

$$C_2\sigma \leq \mathrm{Length}(\mathcal{N}(\phi_\sigma)) \leq C_1\sigma.$$

Proof uses a combination of pseudodifferential techniques and potential theory near the boundary (construction of exponentially accurate quasimodes),

Nodal length of Steklov eigenfunctions on real analytic surfaces

Part (i) of the open problem has been recently solved for real analytic surfaces:

Theorem (P.–Sher–Toth '15) Let M be a real-analytic Riemannian surface. Then there exist positive constants C_1, C_2 such that:

$$C_2\sigma \leq \mathrm{Length}(\mathcal{N}(\phi_\sigma)) \leq C_1\sigma.$$

Proof uses a combination of pseudodifferential techniques and potential theory near the boundary (construction of exponentially accurate quasimodes), and Almgren's frequency function methods for harmonic functions in the interior.

Nodal length of Steklov eigenfunctions on real analytic surfaces

Part (i) of the open problem has been recently solved for real analytic surfaces:

Theorem (P.–Sher–Toth '15) Let M be a real-analytic Riemannian surface. Then there exist positive constants C_1, C_2 such that:

$$C_2\sigma \leq \mathrm{Length}(\mathcal{N}(\phi_\sigma)) \leq C_1\sigma.$$

Proof uses a combination of pseudodifferential techniques and potential theory near the boundary (construction of exponentially accurate quasimodes), and Almgren's frequency function methods for harmonic functions in the interior.

Thank you for your attention