Estimates of shape derivatives around nonsmooth domains

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Shape optimization and spectral geometry
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Introduction and notation

- Goal: find estimates on the shape derivatives $E'(\Omega_0), E''(\Omega_0)$ [Joint work with Jimmy Lamboley and Arian Novruzi] where

$$E(\Omega) = \int_\Omega K(x, u_\Omega(x), \nabla u_\Omega(x))dx, \ \Omega \subset B(0,R) \subset \boldsymbol{R}^d$$

$$K(x,u,q) = \sum_{i,j=1}^{d} \alpha_{ij}(x) q_i q_j + \alpha_0(x) u^2 + \sum_i \beta(x) q_i + \gamma(x) u + \delta(x),$$

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$$Lu_\Omega = f \ in \ \Omega, \ u_\Omega \in H^1_0(\Omega),$$

L elliptic with $W^{2,\infty}$-coefficients and $f \in W^{2,\infty}(B(0,R))$,

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- **semi-convex:** Lipschitz + uniform exterior sphere condition.

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- How much can one improve these continuity property ?

A first result in this context (J. Lamboley, A. Novruzi, M.P.)

Theorem 1. Let $\Omega_0 \subset \boldsymbol{R}^d$, bounded and convex.

- Then, $\theta \in \Theta \to E\big((I + \theta)(\Omega)\big)$ is twice differentiable around $\theta = 0$.

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$$|E''(\Omega_0)(\xi,\xi)| \leq C\|\xi\|^2_{W^{1-\frac{1}{2r},2r}(\partial\Omega_0)}, \quad |E''(\Omega_0)(\xi,\xi)| \leq C\|\xi\|^2_{H^s(\partial\Omega_0)}.$$

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- If moreover $E(\Omega) = \int_\Omega (Lu_\Omega)u_\Omega$ with $L = L^*$, then the above inequalities hold with $r = 1$ and $s = 1/2$:

$$|E'(\Omega_0)\xi| \leq C\|\xi\|_{L^1(\partial\Omega_0)}, \;\; |E''(\Omega_0)(\xi,\xi)| \leq C\|\xi\|^2_{H^{1/2}(\partial\Omega_0)}.$$

And these last estimates are sharp.

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- Extensions to Lipschitz domains
- Some ideas of the proof: mainly relies on the regularity of ∇u_{Ω_0} in Ω_0 + a trace theorem.

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- This shape derivative estimate is e.g. used to prove the following: let $\Omega_0 \subset \boldsymbol{R}^2$ be a solution of the following minimization problem where $R : \boldsymbol{R}^2 \to \boldsymbol{R}$ is smooth and $D_{a,b} = \{x \in \boldsymbol{R}^2;\ 0 < a \leq |x| \leq b < +\infty\}$:

$$(\mathbf{P})\ \min\{R(E(\Omega), |\Omega|) - P(\Omega); \Omega \textit{ convex}, \Omega \subset D_{a,b}\}.$$

Then, Ω_0 is polygonal inside $D_{a,b}$.

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  Then, Ω_0 is polygonal inside $D_{a,b}$.
- If $\partial\Omega_0 \cap \partial D_{a,b} = \{$ finite number of points $\}$, then, Ω_0 is a polygon. In general, the number of extremal points of Ω_0 in the interior of $D_{a,b}$ is finite.

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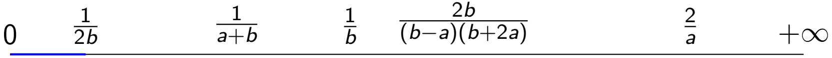
- One purely geometric example:

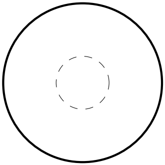
$$\min\{\lambda|\Omega| - P(\Omega),\ \Omega \ convex,\ \Omega \subset D_{a,b}\}.$$

The optimal shapes according to λ for

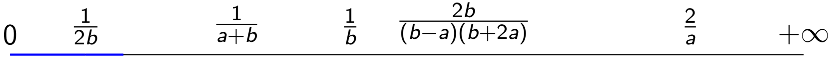
$$\min\{\lambda|\Omega| - P(\Omega),\ \ \Omega\ convex,\ \Omega \subset D_{a,b}\}$$

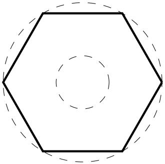
Results from a detailed analysis
by Ch. Bianchini and A. Henrot,
Case $a = 1, b = 3$





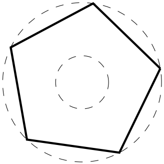
$$0 \leq \lambda < \frac{1}{6}$$





$$0.1792 < \lambda < 0.1847$$

$$0 \quad \frac{1}{2b} \quad \frac{1}{a+b} \quad \frac{1}{b} \quad \frac{2b}{(b-a)(b+2a)} \quad \frac{2}{a} \quad +\infty$$



$$0.1847 < \lambda < 0.19506$$

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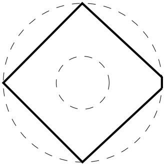
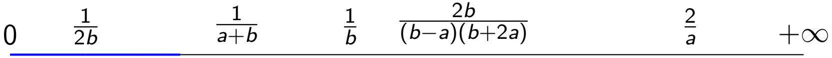
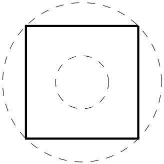
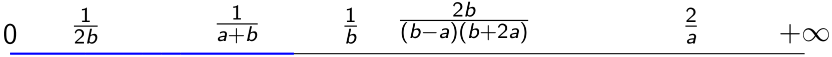


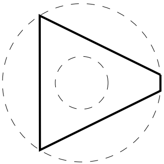
$$0.19506 < \lambda < 0.19525$$



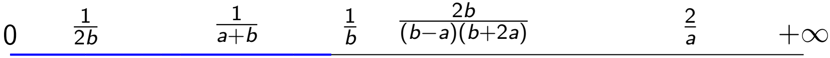


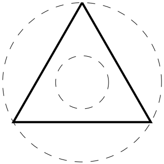
$$0.19525 < \lambda < 0.2187$$



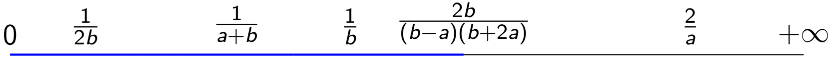


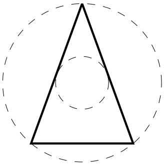
$$0.2187 < \lambda < 0.2222$$



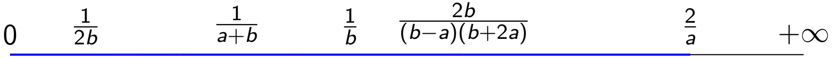


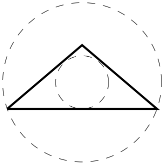
$$0.2222 < \lambda < 0.3080$$



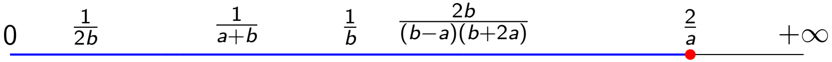


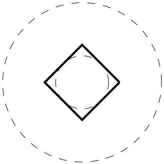
$$0.3080 < \lambda < 0.6$$



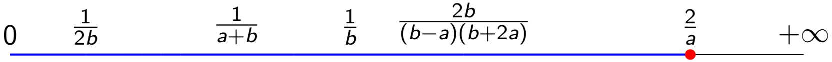


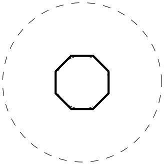
$$0.6 < \lambda < 2$$





$\lambda = 2$





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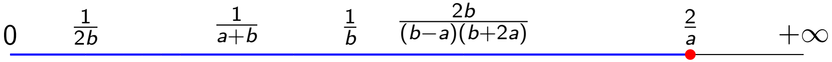

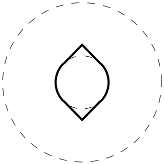

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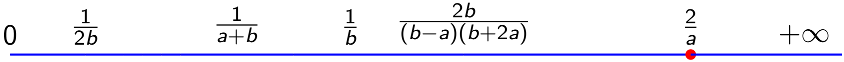

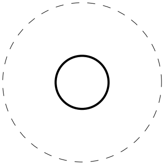

$$2 < \lambda < +\infty$$

Motivations and applications: an analytical result

$$(\mathbf{P}) \ \min\{R(E(\Omega), |\Omega|) - P(\Omega); \Omega \ \textit{convex}, \ \Omega \subset D_{a,b}\}.$$

- The "polygonal result" is a consequence of a more general result in an analytic framework.

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- The "polygonal result" is a consequence of a more general result in an analytic framework.
- Recall that any convex $\Omega \subset \boldsymbol{R}^2$ around the origin may be represented in polar coordinates as

$$\Omega = \Omega_u := \{(\rho, \theta) \in (0, \infty) \times [0, 2\pi];\ \rho = 1/u(\theta),\ u\ 2\pi - periodic\}.$$

$$[\Omega_u\ convex]\ \Leftrightarrow\ [u + u'' \geq 0],$$

$$[\Omega_u\ polygon]\ \Leftrightarrow\ [u + u'' = finite\ sum\ of\ Dirac\ masses].$$

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- Then: $R(E(\Omega_u),|\Omega_u|)-P(\Omega_u)=j(u)$

where $j:W^{1,\infty}(\mathbb{S}^1)\to \boldsymbol{R}$.

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- The geometric problem

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may be rewritten

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- **Theorem.** Let u_0 be a solution of Problem **(j)**. Assume that for some $s \in [0,1)$

$$\forall v \in W^{1,\infty}(\mathbb{S}^1),\ \ j''(u_0)(v,v) \leq -\alpha\|v\|^2_{H^1(\mathbb{S}^1)} + \beta\|v\|^2_{H^s(\mathbb{S}^1)}.$$

  Then , $u_0 + u_0''$ is a finite sum of Dirac masses inside $[a < \frac{1}{u_0} < b]$. And therefore the solution Ω_{u_0} is polygonal inside the annulus.

Motivations and applications: using the shape derivative estimates

- The functional $j(u) = R(E(\Omega_u), |\Omega_u|) - P(\Omega_u)$ does satisfy the "concavity" assumption around u_0 with $u_0 + u_0'' \geq 0$:

$$(*)\ \forall v \in W^{1,\infty}(\mathbb{S}^1),\ \ j''(u_0)(v,v) \leq -\alpha \|v\|^2_{H^1(\mathbb{S}^1)} + \beta \|v\|^2_{H^s(\mathbb{S}^1)}.$$

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- What happens in higher dimensions?

Same problem in any dimension d

- Again the model problem $(D_{a,b} = \{x \in \boldsymbol{R}^d; a \leq |x| \leq b\})$

$$(\mathbf{P}) \ \min\{R(E(\Omega), |\Omega|) - P(\Omega); \Omega \ \mathit{convex}, \partial\Omega \subset D_{a,b}\},$$

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where $j : W^{1,\infty}(\mathbb{S}^{d-1}) \to \boldsymbol{R}$ is C^2 and thanks to Theorem 1

$$j''(u)(v, v) \leq -\alpha \|v\|^2_{H^1(\mathbb{S}^{d-1})} + \beta \|v\|^2_{H^s(\mathbb{S}^{d-1})}, s \in (1/2, 1).$$

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- Can one say more??? (Could be useful for several open problems)

Extension to Lipschitz boundary-The exterior problem

- Replace the "interior Dirichlet energy" $E(\Omega)$ by the "exterior Dirichlet energy" $E_e(\Omega) = \int_{\Omega_e} |\nabla U_\Omega|^2$ where $\Omega_e = \boldsymbol{R}^d \setminus \Omega$ and

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- What is the regularity of $E_e'(\Omega)$ when Ω is convex? What is the regularity of U_Ω? Here Ω_e is merely Lipschitz.
- Similarly, what is the regularity of u_Ω when Ω is Lipschitz?

Regularity of u_Ω, U_Ω

$$\begin{cases} Lu_\Omega = f \ in\ \Omega, u_\Omega = 0 \ on\ \partial\Omega \\ -\Delta U_\Omega = f \ in\ \Omega_e,\ U_\Omega = 0 \ on\ \partial\Omega,\ \ support(f) \subset B(0,R), R\ large. \end{cases}$$

- [Jerison-Kenig '95] Assume Ω is Lipschitz and $L = -\Delta$. Then, $\exists\, p_1$ such that for all $p \in (p_1', p_1)$

$$\|u_\Omega\|_{W^{1,p}(\Omega)} \leq c\|f\|_{W^{-1,p}(\Omega)},$$

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$$p_1 = 4+\eta \ if\ d=2,\ \ p_1 = 3+\eta \ if\ d \geq 3,\ \ 0<\eta=\eta(data).$$

The regularity $W^{1,p}, p < p_1$ is optimal even for f smooth.

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- If Ω is semi-convex, $W^{1,p}$-estimate valid for all $p \in [1, \infty)$ for u_Ω. [S.J. Fromm, H. Jia, D. Li, L. Wang].
  If moreover $f \in L^\infty(\Omega)$, then $u_\Omega \in W^{1,\infty}(\Omega)$.
  semi-convex: Lipschitz + uniform exterior ball condition.

Regularity of the quadratic energies for Lipschitz domains

Theorem 2. Let $\Omega_0 \subset \boldsymbol{R}^d$, bounded and Lipschitz.

- Then, $\theta \in \Theta \to E\big((I+\theta)(\Omega)\big),\ E_e\big((I+\theta)(\Omega)\big)$ are twice differentiable around $\theta = 0$.

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- If Ω_0 is bounded and convex or only semi-convex, then:
  - nothing better is valid for E_e
  - one may take $r_1 = 1$ (i.e. valid for all $r > 1$)
  - one may even take $r = 1$ if $E(\Omega) = \int_\Omega (Lu_\Omega)u_\Omega,\ L = L^*$.

Some ideas of the proof of the estimates on E', E''

- We take the model energy, with a good elliptic operator L

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- $\Rightarrow \theta \to E(\Omega_\theta)$ is C^2 around 0.

Differentiability of $\widehat{u}_\theta$

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- We do not need regularity of Ω_0, only regularity of f and of the coefficients of L ($W^{2,\infty}$)

How to estimate $E'(\Omega_0)\xi, E''(\Omega)(\xi,\xi)$?

Case : $E(\Omega) = \int_\Omega |\nabla u_\Omega|^2$, $L = -\Delta$, Ω regular

- $E'(\Omega)\xi = \int_{\partial\Omega_0} |\nabla u_0|^2 \xi \cdot \nu, \quad u_0 := u_{\Omega_0}.$

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- Moreover $Q_2(\varphi,\varphi) = -\int_{\partial\Omega_0} V\partial_\nu V + \varphi^2[2f\partial_\nu u_0 + H(\partial_\nu u_0)^2]$, where H is the mean curvature of Ω_0 and

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- We forget this !

How to estimate $E'(\Omega_0)\xi, E''(\Omega)(\xi,\xi)$?

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- With $u_0 := u_{\Omega_0}$ and $\xi \in \Theta$

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where $'_0$ denotes the derivation in the ξ-direction at $\theta = 0$ and

$$M_0'\xi = -D\xi - D\xi + trace(D\xi)I.$$

How to estimate $E'(\Omega_0)\xi, E''(\Omega)(\xi,\xi)$?

- Recall

$$E(\Omega_\theta) = \int_{\Omega_0} \nabla\widehat{u}_\theta \cdot M_\theta \cdot \nabla\widehat{u}_\theta, \;\; M_\theta = [I + D\theta]^{-1^t}[I+D\theta]^{-1} det(I+D\theta).$$

- With $u_0 := u_{\Omega_0}$ and $\xi \in \Theta$

$$E'(\Omega_0)\xi = \int_{\Omega_0} 2\nabla u_0 \cdot \nabla\widehat{u}_0' + \nabla u_0 \cdot M_0' \cdot \nabla u_0,$$

where $'_0$ denotes the derivation in the ξ-direction at $\theta = 0$ and

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- If we know $u_0 \in W^{1,p}(\Omega)$ with $p > 2$, then

$$\left|\int_{\Omega_0} \nabla u_0 \cdot M_0' \cdot \nabla u_0\right| \le c\int_{\Omega_0} |\nabla u_0|^2 |D\xi| \le c\|u_0\|^2_{W^{1,p}(\Omega_0)}\|\xi\|_{W^{1,\frac{p}{p-2}}(\Omega_0)}.$$

Estimate of $\int_{\Omega_0} \nabla u_0 \cdot \nabla \widehat{u}_0'$

- Solve the dual problem on the Lipschitz domain Ω_0

$$L^* \phi = -\Delta u_0, \quad \phi \in H_0^1(\Omega_0).$$

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- By the $[W^{-1,p} \to W^{1,p}]$-Jerison-Kenig type regularity result applied to L, for all $p \in (p_1', p_1)$, $u_0 \in W^{1,p}(\Omega_0)$, and when then applied to L^*:
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- $Lu_{\Omega_\theta} = f \;\Rightarrow\; \int_{\Omega_\theta} \phi \circ (I+\theta)^{-1}\, L(u_{\Omega_\theta}) = \int_{\Omega_\theta} \phi \circ (I+\theta)^{-1} f$ and with $[L \text{ on } \Omega_\theta] \mapsto L_\theta \text{ on } \Omega_0$
$$\int_{\Omega_0} \phi L_\theta \widehat{u}_\theta = \int_{\Omega_0} \phi\, \widehat{f}_\theta, \quad \widehat{f}_\theta = [f \circ (I+\theta)] det(I + D\theta)$$
$$\int_{\Omega_0} \phi\, L\widehat{u}_0' = \int_{\Omega_0} \phi\, [\widehat{f}_0' - L_0' u_0] \leq c\|\phi\|_{W^{1,p}}[1 + \|u_0\|_{W^{1,p}}]\|\xi\|_{W^{1,\frac{p}{p-2}}}.$$

Estimate of $E'(\Omega_0)\xi$

- Recalling $E'(\Omega_0)\xi = \int_{\Omega_0} 2\nabla u_0 \cdot \nabla \widehat{u}_0' + \nabla u_0 \cdot M_0' \cdot \nabla u_0$, we have

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- $p_1 = 3 + \eta$ if Ω Lipschitz ($p_1 = 4 + \eta$ if $d = 2$).
  - Valid with $p_1 = +\infty$ if Ω semi-convex (i.e. valid for any $p \in [1, \infty)$.
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- End of the story: the fact that $E'(\Omega_0)\xi$ depends only on the trace of ξ on $\partial\Omega_0$ + a trace lemma for Ω Lipschitz:

$$\inf\left\{\|\widehat{\xi}\|_{W^{1,q}(\Omega)}; \widehat{\xi} \in W^{1,\infty}(\Omega), \widehat{\xi} = \xi \text{ on } \partial\Omega\right\} \leq c\|\xi\|_{W^{1-\frac{1}{q},q}(\partial\Omega)},$$

for $q \in [1, \infty]$.

Estimate of $E''(\Omega_0)(\xi, \xi)$

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- By the same computation as before, with ϕ replaced by $\widehat{u}'_0$
$$\|\widehat{u}'_0\|^2_{H^1_0} \leq \int \widehat{u}'_0 (L\widehat{u}'_0) = \int \widehat{u}'_0 [\widehat{f}'_0 - L'_0 u_0] \leq c\|\widehat{u}'_0\|_{H^1_0} \|\xi\|_{W^{1,\frac{2p}{p-2}}} (1 + \|u_0\|_{W^{1,p}}).$$

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- Differentiating twice $\int_{\Omega_0} L_\theta \widehat{u}_\theta = \int_{\Omega_0} \phi \widehat{f_\theta}$ leads to

$$\int_{\Omega_0} \phi L \widehat{u}''_0 = \int_{\Omega_0} \phi \left(\widehat{f}''_0 - 2L'_0 \widehat{u}'_0 + L''_0 u_0 \right),$$

and we estimate $\widehat{u}''_0$ by duality as for $\widehat{u}'_0$ up to the change

$$W^{1,\frac{p}{p-2}} \to W^{1,\frac{2p}{p-2}}$$

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  it says:
- If $d=2$, then optimal shapes have the polygonal property.
- If $d\geq 3$, estimates of the derivatives of $E_e(\Omega_0)$ are not sufficient to conclude.
  $\rightarrow$ Open problem !.