Local density of states and the spectral function for almost-periodic operators

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Joint work with R.Shterenberg (UAB)

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$$H = -\Delta + b$$

with $b \in \mathrm{USB}(\mathbb{R}^d)$. Let $E(\lambda; H)$ be the spectral projection of H and $e(\lambda; H; \mathbf{x}, \mathbf{y})$ be its integral kernel (also called the *spectral function*). We put $N(\lambda; \mathbf{x}; H) := e_\lambda(\mathbf{x}, \mathbf{x})$ and call $N(\lambda; \mathbf{x})$ the *Local Density of States* of H.

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First – a Conjecture (even two of them!)

Conjecture 1

As $\lambda \to \infty$ we have the complete asymptotic expansion:

$$N(\lambda; \mathbf{x}) \sim \lambda^{d/2}\left(C_d + \sum_{j=1}^{\infty} a_j(\mathbf{x})\lambda^{-j}\right)$$

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in the sense that for each $L \in \mathbb{N}$ one has

$$N(\lambda;\mathbf{x}) = \lambda^{d/2}\Big(C_d + \sum_{j=1}^{L} a_j(\mathbf{x})\lambda^{-j}\Big) + R_L(\lambda;\mathbf{x})$$

with $R_L(\lambda;\mathbf{x}) = o(\lambda^{\frac{d}{2}-L})$. Here, $C_d = \frac{1}{2^d \pi^{d/2}\Gamma(1+d/2)}$ is the Weyl constant.

If this conjecture holds, the coefficients $a_j(\mathbf{x})$ must be real numbers which depend on the potential b and its derivatives at $\mathbf{x}$. They can be calculated using the heat kernel invariants, computed by Polterovich, Hitrik-Polterovich, and Korotyaev-Pushnitski. For example,

$$a_1(\mathbf{x}) = -\frac{dw_d}{2(2\pi)^d} b(\mathbf{x})$$

and

$$a_2(\mathbf{x}) = \frac{d(d-2)w_d}{8(2\pi)^d}[b^2(\mathbf{x}) - \frac{\Delta b(\mathbf{x})}{3}],$$

where w_d is the volume of the unit ball in $\mathbb{R}^d$.

Conjecture 2

Suppose, two potentials $b_1, b_2 \in \mathrm{USB}(\mathbb{R}^d)$ coincide in a neighbourhood of **x**. Then

$$N(\lambda, \mathbf{x}; H_1) - N(\lambda, \mathbf{x}; H_2) = O(\lambda^{-\infty}).$$

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So far these conjectures (they are equivalent!) have been proved if b has compact support (Popov, Shubin, Vainberg) using the wave equation method.
Even when $d = 1$ this is an open (and apparently difficult) question.

We now restrict ourselves to the case when b is almost-periodic (and real); for simplicity we restrict ourselves to the periodic or quasi-periodic case; by quasi-periodic we mean

$$b(\mathbf{x}) = \sum_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \hat{b}(\boldsymbol{\theta}) e^{i\boldsymbol{\theta}\mathbf{x}},$$

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where $\boldsymbol{\Theta}$ is a finite set of frequencies. We can also deal with the 'proper' almost-periodic case, but the results contain too many technical assumptions.

The Pastur-Shubin Theorem implies that for a.e. λ we have:

$$N(\lambda) = \mathbf{M}_{\mathbf{x}} e_\lambda(\mathbf{x}, \mathbf{x}) = \lim_{L \to \infty} \frac{\int_{[-L,L]^d} e_\lambda(\mathbf{x}, \mathbf{x}) d\mathbf{x}}{(2L)^d},$$

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in the sense that for each $L \in \mathbb{N}$ one has

$$N(\lambda; \mathbf{x}) = \lambda^{d/2}\Big(C_d + \sum_{j=1}^{L} a_j(\mathbf{x})\lambda^{-j}\Big) + R_L(\lambda; \mathbf{x}) \tag{2}$$

with $R_L(\lambda; \mathbf{x}) = o(\lambda^{\frac{d}{2}-L})$

Asymptotic expansion (1) (uniform in **x**) would imply the complete asymptotic expansion of $N(\lambda)$ as $\lambda \to \infty$:

$$N(\lambda) \sim \lambda^{d/2}\Big(C_d + \sum_{j=1}^{\infty} e_j \lambda^{-j}\Big), \tag{3}$$

meaning that for each $K \in \mathbb{N}$ one has

$$N(\lambda) = \lambda^{d/2}\Big(C_d + \sum_{j=1}^{K} e_j \lambda^{-j}\Big) + R_K(\lambda) \tag{4}$$

with $R_K(\lambda) = o(\lambda^{\frac{d}{2}-K})$ and $e_j = \mathbf{M}a_j(\mathbf{x})$.

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In particular, Yu.Karpeshina showed that when $d = 3$, formula (4) is valid with $K = 1$ and $R(\lambda) = O(\lambda^{-\delta})$ with some small positive δ and $R(\lambda) = O(\lambda^{\frac{d-3}{2}} \ln \lambda)$ when $d > 3$.

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If b is almost-periodic and $d = 1$, formula (3) was proved by Savin (1988). For $d \geq 2$, (4) is known only with $K = 0$ and $R(\lambda) = O(\lambda^{\frac{d-2}{2}})$ (Shubin, 1987).

Theorem. (R.Shterenberg,LP; 2009 if $d = 2$, 2012 if $d \geq 3$)

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Condition A. Suppose that $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_d \in Z(\boldsymbol{\Theta})$. Then $Z(\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_d)$ is discrete.

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This condition can be reformulated like this: suppose, $\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_d\in Z(\boldsymbol{\Theta})$.Then either $\{\boldsymbol{\theta}_j\}$ are linearly independent, or $\sum_{j=1}^d n_j\boldsymbol{\theta}_j = 0$, where $n_j\in\mathbb{Z}$ and not all n_j are zeros.

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Theorem. (R.Shterenberg,LP; 2012)

Formula (3) *is valid for quasi-periodic b satisfying Condition A.*

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Otherwise, only 1-term asymptotics was known for $N(\lambda; \mathbf{x})$ and upper and lower bounds for $e_\lambda(\mathbf{x}, \mathbf{y})$.

Theorem. (R.Shterenberg,LP; 2014)

Suppose that H is either smooth periodic, or quasi-periodic and satisfies Condition A. Then

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and for $\mathbf{x} \neq \mathbf{y}$ *we have:*

$$e_\lambda(\mathbf{x},\mathbf{y}) \sim \cos(\lambda^{1/2}|\mathbf{x}-\mathbf{y}|)\sum_{q=0}^{\infty} \acute{a}_q(\mathbf{x},\mathbf{y})\lambda^{(d-1)/4-q/2}$$
$$+ \sin(\lambda^{1/2}|\mathbf{x}-\mathbf{y}|)\sum_{q=0}^{\infty} \grave{a}_q(\mathbf{x},\mathbf{y})\lambda^{(d-1)/4-q/2},$$

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assuming that $\mathbf{x}-\mathbf{y}$ *belongs to a generic set of directions.*

Actually, we prove the following formula:

$$e_\lambda(\mathbf{x},\mathbf{x}) = C_d\lambda^{d/2} + \sum_{p=0}^{d-1}\sum_{j=-d+1}^{M} e_{j,p}(\mathbf{x})\lambda^{-j/2}(\ln\lambda)^p + o(\lambda^{-M/2}),$$

but most of the coefficients turn out to be zero due to Hitrik-Polterovich.

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3. H_2 is close to H_1, for example $||H_1 - H_2|| < \lambda^{-N}$ where N is arbitrary large (but fixed) number;

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3. H_2 is close to H_1, for example $||H_1 - H_2|| < \lambda^{-N}$ where N is arbitrary large (but fixed) number; in fact, we will need a stronger bound $||(H_1 - H_2)(-\Delta + I)^s|| < \lambda^{-N}$, where s is fixed.

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1. Is it true that $N(H, \lambda) = N(H_1, \lambda)$?
2. Is it true that $N(H_2, \lambda) - N(H_1, \lambda)$ is small?

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In particular, $N(\lambda; H) = N(\lambda; U^{-1}HU) = N(\lambda; H_1)$, since U is a unitary operator with almost-periodic coefficients and so belongs to the von Neumann algebra.

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This monotonicity follows from the 'box' definition of $N(H, \lambda)$, but this requires H to be differential, and our operators H_1 and H_2 are pseudo-differential. Thus, we again have to use (a variational version of) the definition using von Neumann algebras.

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Well, $\delta_\mathbf{x}$ does not belong to $L_2(\mathbb{R}^d)$, but $E_\lambda(H)\delta_\mathbf{x}$ does!

Lemma.

$E_\lambda(H)\delta_{\mathbf{x}_0} \in L_2(\mathbb{R}^d)$ *and for large* λ *we have*

$$||E_\lambda(H)\delta_{\mathbf{x}_0}||_2 = O(\lambda^{d/4}).$$

Lemma.

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Proof.

We have:

$$\begin{aligned} ||E_\lambda(H)\delta_{\mathbf{x}_0}||_2^2 = \int_{\mathbb{R}^d} |e(\lambda; H; \mathbf{x}_0, \mathbf{y})|^2 d\mathbf{y} = \\ \int_{\mathbb{R}^d} e(\lambda; H; \mathbf{x}_0, \mathbf{y}) e(\lambda; H; \mathbf{y}, \mathbf{x}_0) d\mathbf{y} \\ = e(\lambda; H; \mathbf{x}_0, \mathbf{x}_0) = O(\lambda^{d/2}). \end{aligned} \tag{5}$$

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Lemma.

Suppose, $||H_1 - H_2|| < \varepsilon$. Then for each f and each $\delta \geq \varepsilon$ we have (take $\delta = \varepsilon^{1/2}$):

$$||(E_\lambda(H_2) - E_\lambda(H_1))f|| \leq 2||E([\lambda - \delta, \lambda + \delta]; H_2)f|| + 2\pi\varepsilon\delta^{-1}||E((-\infty, \lambda]; H_2)f|| + 2\pi\varepsilon\delta^{-1}||f||.$$

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Lemma.

Suppose, $||H_1 - H_2|| < \varepsilon$. Then for each f and each $\delta \geq \varepsilon$ we have (take $\delta = \varepsilon^{1/2}$):

$$||(E_\lambda(H_2) - E_\lambda(H_1))f|| \leq 2||E([\lambda - \delta, \lambda + \delta]; H_2)f|| \\ + 2\pi\varepsilon\delta^{-1}||E((-\infty, \lambda]; H_2)f|| + 2\pi\varepsilon\delta^{-1}||f||.$$

We want to take $f = \delta_{\mathbf{x}}$, so this estimate is not good enough.

Lemma.

Suppose, $H_2 + AI > 0$ *and* $||(H_1 - H_2)(H_2 + AI)^s|| < \varepsilon$*. Then for each* f *and each* $\delta \geq \varepsilon$ *we have:*

$$||E_\lambda(H_2)f - E_\lambda(H_1)f|| \leq 2||E([\lambda - \delta, \lambda + \delta]; H_2)f||$$
$$+ 2\pi\varepsilon\delta^{-1}||E((-\infty, \lambda]; H_2)f|| + 2\pi\varepsilon\delta^{-1}||(H_2 + AI)^{-s}f||.$$