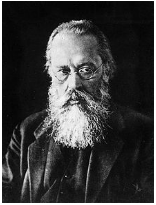

Steklov spectral inequalities through quasiconformal mapping

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I. Survey of the Steklov eigenvalue problem

[Girouard & Polterovich *Spectral geometry of the Steklov problem*, 2014]

Vibrating simply conn. free membrane $\Omega \subset \mathbb{R}^2$, mass concentrated on boundary. Transverse displacement $u(x, y)$ has Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^2 \, dA}{\int_{\partial\Omega} u^2 \, ds}$$

where ds =arclength element.

Euler-Lagrange gives **Steklov eigenvalue problem**

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \sigma u & \text{on } \partial\Omega \end{cases}$$

Eigenfunction u is harmonic, and Steklov eigenvalues are

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty.$$

Note $u_0 \equiv$ const.

Example — unit disk

$u = r^k \cos k\theta$ and $u = r^k \sin k\theta$ e.g. $k = 2$

$$\frac{\partial u}{\partial n} = ku \quad \Longrightarrow \quad \text{spectrum } \{0, 1, 1, 2, 2, 3, 3, \ldots\}$$

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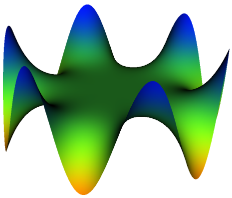
e.g. $k = 6$, whispering gallery:

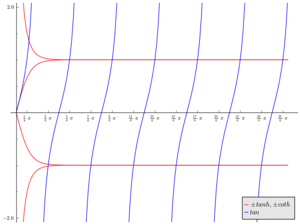


Example — square $(-1,1)\times(-1,1)$

Separation of variables: find $u = xy$ with $\sigma = 1$. Also
$u = \sin(\alpha x)\sinh(\alpha y)$, where α is root of $\boxed{\tan\alpha = \tanh\alpha}$

$$\Longrightarrow \quad \text{eigenvalue} \quad \sigma = \alpha\coth\alpha.$$

Find cos, cosh solutions too. Each intersection in plot below (from G&P 2014) gives a double eigenvalue:



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$u = \sin(\alpha x)\sinh(\alpha y)$, where α is root of $\boxed{\tan\alpha = \tanh\alpha}$
$\implies$ eigenvalue $\sigma = \alpha\coth\alpha$.

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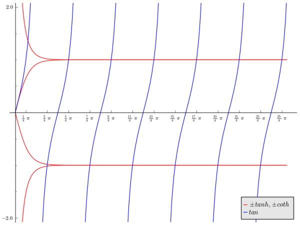

How prove L^2-*boundary completeness* of eigenfunctions?! G&P achieve it by reducing to 1-d beam equation on edges of the square.

Rectangle — boundary completeness of the separated eigenfunctions: Girouard, Polterovich and Savo (in preparation)

II. Few lower bounds are known on eigenvalues

Recall variational characterization of eigenvalues via Rayleigh quotient…

(a) Length normalization is useless!

$\sigma_j L$ can be arbitrarily close to 0, for thin rectangle $(0, \pi) \times (0, \epsilon)$:

Intuition: area vanishes as $\epsilon \to 0$, but boundary length does not.

(Precisely, use $u(x, y) = \sin(jx)$ as boundary-orthogonal trial functions in Rayleigh quotient. Get $\sigma_{j-1} \leq \epsilon j^2/2 \to 0$ as $\epsilon \to 0$.)

(b) Try introducing additional geometric information…

Payne (1970) proved for convex Ω that

$$\sigma_1 \geq (\text{min. curvature on } \partial\Omega).$$

Escobar and Jammes obtained lower bounds of Cheeger type.

III. Upper bounds on eigenvalues, by conformal mapping

First eigenvalue by Weinstock (1954):

$$\boxed{\sigma_1 L \leq 2\pi}$$

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Higher eigenvalues by Hersch–Payne–Schiffer (1975):

$$\boxed{\sigma_j L \leq 2\pi j}$$

Maximizer is disjoint union of j disks (limiting case), proved Girouard–Polterovich (2010).

Goal in this Talk: get better bounds on domains that are unlike disjoint unions of disks?

Spectral sums by quasiconformal transplantation

Theorem (Girouard, Laugesen & Siudeja; ARMA 2015)

Assume $f : \mathbb{D} \to \Omega$ is quasiconformal with complex dilatation μ depending only on the angular variable θ. Then

$$\sum_{j=1}^{n} \sigma_j L \leq \sum_{j=1}^{n} \left\lceil \frac{j}{2} \right\rceil 2\pi\, g$$

for each $n \in \mathbb{N}$, with equality if Ω is a disk.
The geometric factor $g = g(f)$ is explicitly computable.

Intuition: $g \geq 1$ with equality for disk, and so

g penalizes deviation from disk

Questions... Why care about eigenvalue sums?
How is the theorem proved? What does it say for conformal maps? for starlike domains? How does it compare with Hersch–Payne–Schiffer?

Eigenvalue sums are powerful...

Corollary

The following spectral functionals attain their maximum when Ω is a disk:

$$\sigma_1 L/g, \quad (\sigma_1^s + \cdots + \sigma_n^s)^{1/s} L/g, \quad \sqrt[n]{\sigma_1 \cdots \sigma_n}\, L/g$$

for each $0 < s \leq 1$.
For $s < 0 < t$, the partial sums of the spectral zeta function and heat trace, namely

$$\sum_{j=1}^{n} (\sigma_j L/g)^s \qquad \text{and} \qquad \sum_{j=1}^{n} \exp(-t\sigma_j L/g),$$

are minimal when Ω is a disk.

Proof: Spectral sums theorem + Hardy–Littlewood–Pólya majorization

Proof of Theorem

• Obtain *trial functions* on Ω by transplanting trigonometric eigenfunctions from disk, by quasiconformal map f

• Impose *orthogonality* of trial functions on boundary of Ω by "angular uniformization" of the circle

• Apply *Rayleigh variational characterization for eigenvalue sum* (taking minimum over orthogonal trial functions)

• Invoke *quasi-invariance of Dirichlet integral*

$$\int_\Omega |\nabla(u \circ f^{-1})|^2 \, dA = \int_{\mathbb{D}} \left\{ a_0 u_r^2 + a_1 \frac{u_\theta^2}{r^2} + a_2 u_r \frac{u_\theta}{r} \right\} r dr d\theta$$

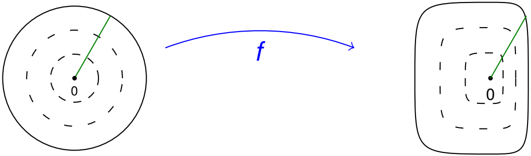
with coefficients

$$a_0 = \frac{|e^{2i\theta} - \mu(re^{i\theta})|^2}{1 - |\mu(re^{i\theta})|^2}, \qquad a_1 = \frac{|e^{2i\theta} + \mu(re^{i\theta})|^2}{1 - |\mu(re^{i\theta})|^2}, \qquad a_2 = \text{stuff.}$$

Here $\mu = \overline{\partial} f / \partial f$ is the complex dilatation (e.g. f conformal, $\mu \equiv 0$)

Is geometric factor g computable?

Starlike domain $\Omega = \{re^{i\theta} : \theta \in [0, 2\pi], 0 \le r < \rho(\theta)\}$



for *radius function* function $\rho(\theta)$. Then $\boxed{g = \sqrt{g_0 g_1}}$ where

$$g_0 = 1 + \frac{1}{2\pi} \int_0^{2\pi} (\log \rho)'(\theta)^2 \, d\theta,$$

$$g_1 = \frac{1}{2\pi L^2} \int_0^{2\pi} \left(\rho(\theta)^2 + \rho'(\theta)^2\right) d\theta \,,$$

Conclusion.

If we know the radius function for a starlike domain, then g is computable.

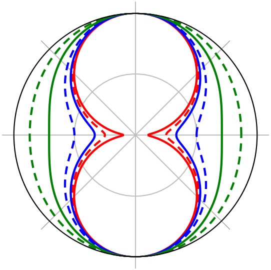
Optimal choice of origin will be center of symmetry, if it exists.

Is g computable for conformal map f?

$$g = \frac{2\pi}{L}\left\{\left(\frac{1}{2\pi}\int_0^{2\pi}|f'(e^{i\theta})|^2\,d\theta\right)^2 - \left|\frac{1}{2\pi}\int_0^{2\pi}|f'(e^{i\theta})|^2 e^{i\theta}\,d\theta\right|^2\right\}^{1/4}$$

Conformal method is suitable on smooth domains.
(If $\partial\Omega$ has corners then $|f'|^2$ can blow up.)

Example — hippopedes (inverted ellipses)



$1-\varepsilon^2$	1/16	1/9	1/4	1/2	3/4	1
numerical g	1.1016	1.0924	1.0692	1.0281	1.0056	1
conformal g	1.6078	1.4302	1.2112	1.0627	1.0115	1
starlike g	1.4909	1.3214	1.1378	1.0366	1.0064	1

On not-too-eccentric hippopedes, the conformal and starlike methods both give better eigenvalue sum estimates than Hersch–Payne–Schiffer (since $g < 1.5$ in those cases).

Future directions

1. What kind of quasiconformal maps have purely angular dilatation? Can one compute the corresponding geometric factor g in practice?

2. Stability for Weinstock's inequality? If $\sigma_1 L$ is close to 2π then must the simply connected domain be close to a disk?
(Done under area normalization by Brasco, De Philippis and Ruffini for general domains.)

3. Hersch–Payne–Schiffer showed how to maximize individual Steklov eigenvalues (answer: union of disks).

What domains maximize *combinations* of eigenvalues?

Disk is known to maximize geometric mean $\sqrt{\sigma_1 \sigma_2}\, L$.

Shape Optimization Problem
What simply connected domain maximizes the arithmetic mean $\frac{1}{2}(\sigma_1 + \sigma_2)L$???

Numerics suggest a stadium-like domain (Siudeja; unpublished).

From sums to heat trace by ***majorization***

If $a_1 \leq a_2 \leq a_3 \leq \cdots$ and $b_1 \leq b_2 \leq b_3 \leq \cdots$ and

$$a_1 + \cdots + a_n \leq b_1 + \cdots + b_n \qquad \forall n \geq 1$$

then

$$C(a_1) + \cdots + C(a_n) \leq C(b_1) + \cdots + C(b_n) \qquad \forall n \geq 1$$

for all concave increasing functions C.
(*Fun exercise.* Prove it for $n = 1, 2$.)

Example:
using $C(z) = -\exp(-zt)$ gives heat trace

Weyl asymptotics

For smooth boundary,

$$\boxed{\sigma_j L = j\pi + O(1)} \qquad \text{as } j \to \infty$$

where L is length of $\partial\Omega$.

Proof: pseudo differential operator techniques for Dirichlet-to-Neumann operator.

Improved error estimate by Rozenbljum says eigenvalues come in pairs, and converge onto disk eigenvalues faster than any power of j:

$$\sigma_{2j} L = 2j\pi + O(j^{-\infty})$$
$$\sigma_{2j-1} L = 2j\pi + O(j^{-\infty})$$

Open problem: prove or disprove Weyl asymptotic $\sigma_j L = j\pi + O(1)$ assuming just Lipschitz boundary.

Inverse spectral problem

Mark Kac

Can you hear the shape of a Steklov membrane?

No counterexamples known for plane domains. (Isospectral non-planar flat surfaces found by Gordon, Herbrich and Webb 2015.)

Cut-and-paste constructions (used for Laplace eigenvalues) fail since the eigenvalue appears in the boundary condition!

What *can* be heard?

Spectrum determines boundary length (by Weyl),
number of boundary components (by Girouard *et al.* 2014).