Optimal compliance problem

Antonin Chambolle, Jimmy Lamboley, Antoine Lemenant, Eugene Stepanov

University Paris Dauphine, CEREMADE

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Statement of the problem

We consider the optimization problem

$$\min \left\{ \mathcal{C}(\Sigma),\ \ \Sigma \in \mathcal{A}(\Omega),\ \ \mathcal{H}^1(\Sigma) \leq L \ \right\}$$

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- $\mathcal{C}(\Sigma)=\displaystyle\int_\Omega f u_\Sigma \quad \textit{where} \quad \left\{\begin{array}{rcll} -\Delta u_\Sigma & = & f & \textit{in}\ \ \Omega\setminus\Sigma \\ u_\Sigma & = & 0 & \textit{on}\ \ \partial\Omega\cup\Sigma \end{array}\right.$

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- $\mathcal{H}^1(\Sigma) = \textit{ length of } \Sigma.$

Questions

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- Geometrical description of the solution :
  - Saturation of the constraint ?
  - Are the optimal sets regular ?
  - Are there loops in the optimal set ?

Our result

About a penalized version

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Theorem

Assume $f \in L^p(\Omega)$ with $p > 2$, and Σ_{opt} is optimal. Then

- *Σ_{opt} contains no closed curves,*
- *Σ_{opt} consists in a finite number of $C^{1,\alpha}$-curves, possibly intersecting at "triple points" where the curves form 120° angles.*

Outline

Average distance problem

Irrigation problem

$$\min\left\{\int_\Omega dist(x,\Sigma)f(x)dx+\lambda\mathcal{H}^1(\Sigma),\ \ \Sigma\in\mathcal{A}(\Omega)\right\}.$$

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- No loop,
- Classification of blow-ups,
- No full-regularity in general (corners),
- Γ-limit of the p-compliance problem when $p\to\infty$.

Mumford-Shah problem

Segmentation

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2\,dx+\int_\Omega (u-g)^2\,dx+\mathcal{H}^1(\Sigma),\right.$$
$$\left.\Sigma\subset\overline{\Omega}\ compact,\quad u\in H^1(\Omega\setminus\Sigma)\right\}$$

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$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega (u-g)^2 \, dx + \mathcal{H}^1(\Sigma), \right.$$
$$\left. \Sigma \subset \overline{\Omega} \textit{ compact}, \quad u \in H^1(\Omega \setminus \Sigma) \right\}$$

- Conjecture (open) : Σ is a finite union of C^1-curves
- Classification of **connected** blow-up limits

A. Bonnet, *On the regularity of edges in image segmentation, Ann. Inst. H. Poincaré Anal. Non Linéaire,* 1996

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Monotonicity formula

Estimate of the energy

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Let $0 \leq r_0 < r_1$ and x_0 such that

$$\forall r \in [r_0, r_1], \quad \Sigma \cap \partial B_r(x_0) \neq \emptyset.$$

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and $\gamma \in (0, 2\pi]$ such that

$$\gamma \geq \sup \left\{ \frac{\mathcal{H}^1(S)}{r} : r \in (r_0, r_1) \text{ and } S \text{ connected} \subset \partial B_r(x_0) \setminus \Sigma \right\},$$

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Lemma

The function

$$r \in [r_0, r_1] \mapsto \frac{1}{r^{\frac{2\pi}{\gamma}}}\left(\int_{B_r(x_0)} |\nabla u_\Sigma|^2 \, dx + C r^{\frac{2}{p'}}\right),$$

is nondecreasing for some $C = C(|\Omega|, p, \|f\|_p, \gamma)$.

Variations of the compliance

Proposition

For every $\Sigma' \in \mathcal{A}(\Omega)$ *satisfying* $\Sigma \Delta \Sigma' \subset B_r(x_0)$ *we have*

$$|\mathcal{C}(\Sigma') - \mathcal{C}(\Sigma)| \leq C \left(r^{\frac{2\pi}{\gamma}} + r^{\frac{2}{p'}} \right)$$

where C *is depending on* Ω, f, γ, p, r_1.

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$$\lambda r \leq \lambda\left(\mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1(\Sigma^r_{opt})\right) \leq \mathcal{C}(\Sigma^r_{opt}) - \mathcal{C}(\Sigma_{opt}) \leq C\left(r^{2-\varepsilon} + r^{\frac{2}{p'}}\right)$$

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Contradiction

Outline

Flatness implies regularity

- Flatness : $\beta_\Sigma(x,r) := \inf\left\{\frac{1}{r}d_H(\Sigma\cap B_r(x), P\cap B_r(x)),\ \text{lines } P\right\}$

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- $\omega_\Sigma(x,r) = \max\left\{\frac{1}{r}\int_{B_r(x)}|\nabla u_{\Sigma'}|^2\,dx,\quad \Sigma'\in\mathcal{A}(\Omega),\quad \Sigma'\Delta\Sigma\subset\overline{B}_r(x)\right\}.$

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- β_Σ and ω_Σ small enough implies

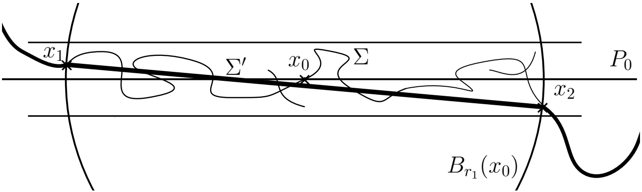

FIGURE : Construction of the competitor Σ'.

Classification of blow-ups

Around $x_0 = 0$:

- $\Sigma_n := \frac{1}{r_n}\Sigma, \quad \Omega_n = \frac{1}{r_n}\Omega,$
- $u_n(x) := r_n^{-\frac{1}{2}} u(r_n x) \in H_0^1(\Omega_n \setminus \Sigma_n),$
- $f_n(x) = r_n^{3/2} f(r_n x).$

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Any blow-up limit (u_0, Σ_0) is one of the following list :

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2. Σ_0 *is a propeller (three half lines meeting by 3 by 120 degree angles) and* u_0 *is a constant on each side.*
3. Σ_0 *is a half-line and* u_0 *is the "Dirichlet-craktip" function* $\sqrt{r/2\pi}\cos(\theta/2)$ *in polar coordinates.*

Perspectives

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- $\max\left\{\lambda_1(\Omega \setminus \Sigma),\ \ \Sigma \in \mathcal{A}(\Omega),\ \ \mathcal{H}^1(\Sigma) \leq L\ \right\}.$