# Eigenvalue Estimates for Quantum Graphs

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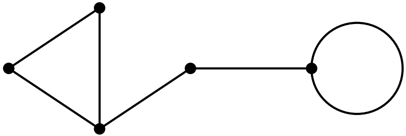

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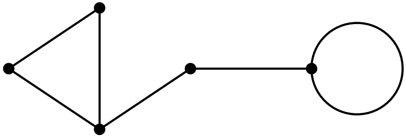

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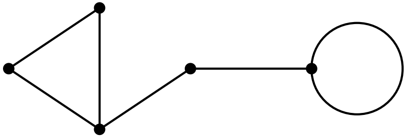

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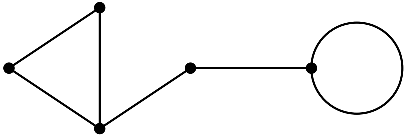

- Two different theories: discrete and continuous
- Discrete graphs: vertices are "adjacent" or "related" if there is an edge connecting them
- Leads to a theory of functions $f : V \to \mathbb{R}$ living on the vertices and discrete difference operators acting on them (e.g. discrete Laplacian)

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- The vertex conditions are generally encoded in the domain of the operator / associated form

# The Kirchhoff Laplacian

$$\begin{aligned} H^1(\Gamma) :=\{u : \Gamma \to \mathbb{R} : u_{|e_j} \in H^1(e_j) \sim H^1(a_j, b_j) \text{ for all edges } e_j \\ \text{and if } e_1 \sim (a_1, b_1) \text{ and } e_2 \sim (a_2, b_2) \text{ share} \\ \text{a common vertex } b_1 \sim a_2, \text{ then } u(b_1) = u(a_2)\} \end{aligned}$$

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- Define a bilinear form $a : H^1(\Gamma) \to \mathbb{R}$ by

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- The associated operator in $L^2(\Gamma)$ is the

Kirchhoff Laplacian $-\Delta_\Gamma$

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  - If $\Gamma$ consists of a single edge connecting two vertices, it is the Neumann Laplacian on an interval
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We will concentrate on $\lambda_1$, i.e. the spectral gap

# Characterisation of $\lambda_1(\Gamma)$

$$\lambda_1(\Gamma) = \inf \left\{ \frac{\|\nabla u\|^2_{L^2(\Gamma)}}{\|u\|^2_{L^2(\Gamma)}} : \, 0 \neq u \in H^1(\Gamma), \, \int_\Gamma u = 0 \right\}$$

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Rayleigh quotient / test function methods are far more powerful in one dimension!

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Simple examples show:

*no upper bound is possible in terms of $L$ alone!*

# An upper bound on $\lambda_1(\Gamma)$

**Theorem (KKMM, 2015)**

*Denote by $E$ the number of edges of $\Gamma$. Then*

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**Open problem**

Identify all possible maximisers.

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There exists a sequence of graphs $\Gamma_n$ ("pumpkin chains") with $D(\Gamma_n) = 1$ and $\lambda_1(\Gamma_n) \to \infty$.

# More bounds on $\lambda_1(\Gamma)$?

**Remark**

$\lambda_1(\Gamma_n) \to \infty$ is a "global" property of $\Gamma_n$: attach a fixed pendant edge $e$ of length $\ell > 0$ to each $\Gamma_n$ to form a new graph $\tilde{\Gamma}_n$, then $\lambda_1(\tilde{\Gamma}_n) \leq \pi^2/\ell^2$ for all $n$. (Surgery principle: attaching the pendant graph $\Gamma_n$ to $e$ can only lower the eigenvalue of $e$!)

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**Theorem (KKMM, 2015)**

*If $\Gamma$ has diameter $D$, $E$ edges and $V \ge 2$ vertices, then*

$$\lambda_1(\Gamma) \le \frac{\pi^2}{D^2}(V+1)^2$$

*and*

$$\frac{\pi^2}{D^2 E^2} \le \lambda_1(\Gamma) \le \frac{4\pi^2 E^2}{D^2},$$

*with equality in the lower bound if $\Gamma$ is a path and in the upper bound if $\Gamma$ is a loop.*

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**Open problem**

Find an optimal upper bound on $\lambda_1$ in terms of $D$ and $V$, and upper and lower bounds in terms of $D$ and $E$, which are optimal for all $E \geq 1$.

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## Summary

| | upper bound | lower bound |
|---|---|---|
| $V$, $E$ | ✗ | ✗ |
| $L$ | ✗ | ✓ |
| $L$, $V$ | ✗ | |
| $L$, $E$ | ✓ | |
| $D$ | ✗ | ✗ |
| $D$, $V$ | ✓ | ✗ |
| $D$, $E$ | ✓ | ✓ |

Thank you for your attention!