Towards the optimal constant for the quantitative isoperimetric inequality in the plane

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Outline

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The quantitative isoperimetric inequality

Let $\Omega \subset \mathbb{R}^N$ with Lebesgue measure $|\Omega|$ and perimeter $P(\Omega)$, its isoperimetric deficit is defined as

$$\delta(\Omega) = \frac{P(\Omega) - P(B)}{P(B)}\,, \quad |B| = |\Omega|\,,\ B = \text{ball}\,,$$

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$$\exists C_N > 0, \text{ such that } \forall \Omega, \quad \lambda(\Omega)^2 \leq C_N\, \delta(\Omega).$$

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A natural question: What is the optimal constant C_N^*?

The optimal constant

The optimal constant C_N^* satisfies

$$\frac{1}{C_N^*} = \inf_{\Omega \neq B} \frac{\delta(\Omega)}{\lambda(\Omega)^2}$$

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This approach is strongly related to the selection principle by M. Cicalese and G.P. Leonardi and we will recover some of their results.

The main result (in the plane)

We can prove:

Theorem (Bianchini, Croce, H. 2015)

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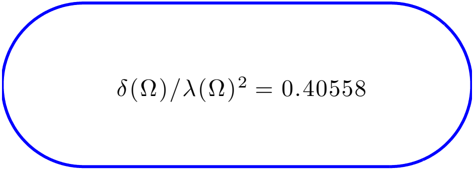
- *Ω^* has (at least) two optimal balls for the Fraenkel asymmetry.*
- *Ω^* is not convex and has at most six connected components.*

Existence and regularity of Ω^* are not new.
We conjecture that Ω^* has exactly two optimal balls for the Fraenkel asymmetry and is actually connected.

The convex case

When restricted to plane convex domains, S. Campi (1992) and A. Alvino, V. Ferone, C. Nitsch (2011) were able to identify the minimizer of $\mathcal{F}(\Omega)$ which is a particular stadium.



Outline

Definition of the rearrangement

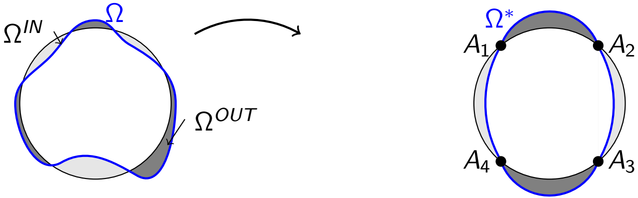


Figure: A set Ω and its rearrangement Ω^*.

The new set Ω^* is obtained by distributing half of the external matter on the north pole and half on the south pole, while half of the internal matter is put on the west pole and half on the east pole, preserving the total lengths of $\partial\Omega^{OUT} \cap \partial B$ and $\partial\Omega^{IN} \cap \partial B$. The boundary of Ω^* is composed of arcs of circle.

Properties of the rearrangement

This rearrangement decreases (asymptotically) the functional $\mathcal{F}(\Omega)$:

Proposition

For every $\alpha > 0$ there exists $\beta > 0$ such that for every Ω with $\lambda(\Omega) \leq \beta$, one has $\mathcal{F}(\Omega^*) \leq \mathcal{F}(\Omega) + \alpha$.

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Then, to study the behaviour of sequences converging to the ball, it suffices to consider their rearrangement

Sequences converging to the ball are not competitive

Theorem

Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$, *be a sequence of sets, such that* $|\Omega_\varepsilon| = \pi = |B|$ *where* B *is a unit ball. Assume that* $|B\Delta\Omega_\varepsilon| = \frac{4\varepsilon}{\pi}$. *Then*

$$\liminf_{\varepsilon\to 0} \mathcal{F}(\Omega_\varepsilon^*) \geq \frac{\pi}{8(4-\pi)} \approx 0.4575.$$

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It follows:

Corollary

Let $\varepsilon > 0$*. Let* Ω_ε *be a sequence of sets converging to a ball* B *such that* $|B\Delta\Omega_\varepsilon| = \frac{4\varepsilon}{\pi}$*. Then*

$$\liminf_{\varepsilon\to 0} \mathcal{F}(\Omega_\varepsilon) = \liminf_{\varepsilon\to 0} \frac{\delta(\Omega_\varepsilon)}{\lambda^2(\Omega_\varepsilon)} \geq \frac{\pi}{8(4-\pi)}.$$

The best sequence

We can be more precise by proving

Theorem

Let $\varepsilon > 0$. Let Ω_ε be any sequence of planar regular sets converging to a ball B. Then

$$\inf \left\{ \liminf_{\varepsilon \to 0} \mathcal{F}(\Omega_\varepsilon) \right\} = \frac{\pi}{8(4-\pi)}.$$

recovering Cicalese-Leonardi's result, and the "best" sequence is given by:

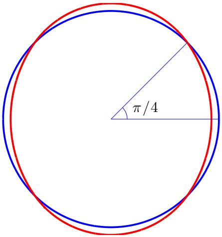
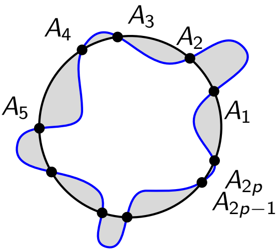


Figure: The "best" sequence converging to the ball

Ingredients: Optimality condition

The optimal ball for a set Ω satisfies some properties



Theorem

Let Ω be a transversal set to an optimal ball B. Then the intersection points $A_i = (x_i, y_i)$ between $\partial\Omega$ and ∂B satisfy

$$x_1 + x_3 + \ldots + x_{2p-1} - (x_2 + x_4 + \ldots + x_{2p}) = 0,$$

$$y_1 + y_3 + \ldots + y_{2p-1} - (y_2 + y_4 + \ldots + y_{2p}) = 0.$$

Location of the optimal ball in a symmetric case

In general, it is not so easy to locate the optimal ball. The following result can be very useful:

Proposition

Assume that $\Omega \subset \mathbb{R}^2$ *has a symmetry axis* Π *and is convex in the perpendicular direction. Then there exists an optimal ball centered on* Π.

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Corollary

Assume that $\Omega \subset \mathbb{R}^2$ *has two (perpendicular) axis of symmetry crossing at* O *and is convex in both directions. Then there exists an optimal ball centered at* O*.*

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This corollary is important to claim that for the rearranged set Ω^*, the optimal ball is still the same.

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Existence of an optimal set

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There exists a set Ω^* *which minimizes* $\mathcal{F}(\Omega) = \frac{\delta(\Omega)}{\lambda^2(\Omega)}$ *among all the subsets of* $\mathbb{R}^2$ *(the ball excluded).*

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Sketch of the proof

Take a minimizing sequence Ω_n, since

- there exists a stadium $\widehat{\Omega}$ for which $\mathcal{F}(\widehat{\Omega}) = 0.40558$,
- for any sequence converging to the ball, $\liminf \mathcal{F}(\Omega_n) \geq 0.45$

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the minimizing sequence cannot converge to the ball.

Since $\lambda(\Omega_n) \leq 2$, $\delta(\Omega_n)$ and then $P(\Omega_n)$ are uniformly bounded. If we can prove that the sequence itself is uniformly bounded (contained in some fixed ball D), existence will classically follow from the compact embedding $BV(D) \hookrightarrow L^1(D)$ and lower-semi continuity of the perimeter.

Number of connected components (1)

To prove that the sequence is uniformly bounded, we will use:

Proposition

Let Ω be a set whose perimeter is less than 20. Then there exists $\widetilde{\Omega}$ composed by at most 7 connected components, such that

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To prove this proposition, we first observe that if a connected set ω is not contained in a unit ball then its perimeter is greater than 4: take the convex hull $\hat{\omega}$ and use

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Thus Ω has at most 4 components not contained in a unit ball and we can replace all other components by balls.

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Since $\delta(\Omega_n) = \frac{P(\Omega_n)-2\pi}{2\pi} \leq 0.4055 * 2^2$, we have $P(\Omega_n) \leq 16.477$, it follows that we can work with a minimizing sequence with at most 7 connected components and by translation, we can assume that all these components lie in a fixed ball of radius 50. Existence follows.

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By looking at the previous proof for the optimal domain itself, we observe that only two balls are needed, thus it has at most 6 connected components.

Number of optimal balls

Our main contribution is to prove

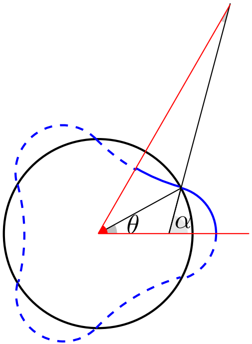
Theorem

Let Ω^* be a minimizer of the functional $\mathcal{F}$. Then, Ω^* has at least two optimal balls realizing the Fraenkel asymmetry.

We argue by contradiction, assuming that there is a unique optimal ball. We distinguish two cases: Ω^* connected or Ω^* disconnected (in that case, it is easy to see that it has only two connected components, one being necessarily a ball).

Idea of the proof

We use the parametrization (N copies of arcs of circle):



Considering all possible values for the parameters α, θ, N we show that we always get a contradiction with one of the following facts:

- $\mathcal{F}(\Omega^*) \leq 0.4055$
- the first order optimality condition which writes $\frac{1}{R_0} + \frac{1}{R_1} = \frac{8\delta}{\lambda}$
- the second order optimality condition.

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What remains open ?

To determine the optimal domain Ω^*, it remains now to prove:

Conjecture

- *The optimal set Ω^* has two perpendicular axes of symmetry.*
- *The optimal set has exactly two optimal balls B_1 and B_2 realizing the Fraenkel asymmetry.*

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With these two properties, we are led to solve a simple minimization problem

The candidate

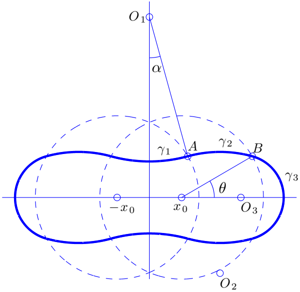

Figure: Parametrization of a mask with α, θ, x_0

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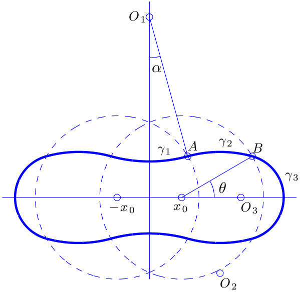

Figure: Parametrization of a mask with α, θ, x_0

By solving the two-dimensional minimization problem, we get:

Conjecture

The solution of the minimization problem is the "mask" with
$\alpha = 0.2686247,\ \theta = 0.5285017,\ x_0 = 0.3940769.$
The value of $\mathcal{F}$ for the set Ω^* is $1/C_2^* = 0.39314$ (and $C_2^* = 2.543625$).