Steklov expanders

On surfaces with large Steklov eigenvalues

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- Upper bounds for Steklov eigenvalues
- Expanders
- Graph-like surfaces
- A spectral comparison inequality
- Surfaces with large σ_1

1. Upper bounds for Steklov eigenvalues

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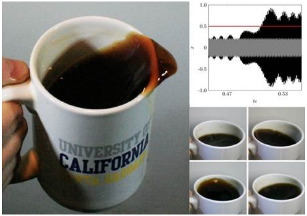
The corresponding eigenfunctions form an orthonormal basis of $L^2(M)$.

Vladimir Steklov

Linear coffee waves

H. C. Mayer and R. Krechetnikov, ***Walking with coffee: Why does it spill?*** (2012)



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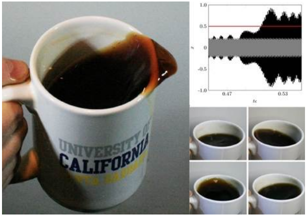

Velocity potential $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}$

$$\begin{aligned} \Delta\Phi &= 0 && \text{inside the mug} \\ \partial_t^2\Phi + g\partial_n\Phi &= 0 && \text{on the free surface} \\ \partial_n\Phi &= 0 && \text{on the sides and bottom.} \end{aligned}$$

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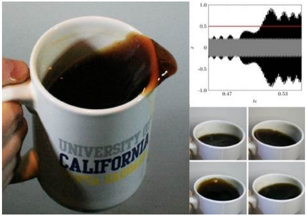

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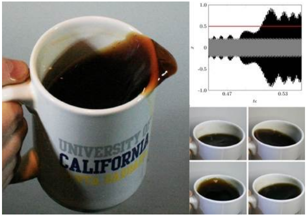

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Winner of the 2012 Ig Nobel prize in physics

http://www.improbable.com/ig/

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A similar result was also announced by M. Karpukhin

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Strategy of the proof

1. Learn about expander graphs
2. Construct surfaces from graphs
3. Prove a spectral comparison theorem

The strategy itself is not new. It was very useful in the context of closed surfaces: Buser, Brooks, Burger, . . .

2. Expander graphs

The Cheeger constant of a finite graph $\Gamma = (V, E)$ is

$$h(\Gamma) := \min \left\{ \frac{|\partial A|}{|A|} \ : A \subset V, 0 < |A| \leq |V|/2 \right\}.$$

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The Cheeger constant measures how good the «edge expansion» of a graph is:

- If $h(\Gamma)$ is small, there is a "bottleneck". The graph is almost disconnected.
- If $h(\Gamma)$ is large, any set $A \subset V$ is well connected to the rest of the graph.

Examples: Circular graphs, complete graphs, bottlenecks.

Let $k \in \mathbb{N}$ and $\epsilon > 0$. A sequence of graphs $\Gamma_n = (V_n, E_n)$ is called a (k, ϵ) **family of expanders** if

- Each Γ_n is a regular graph of degree k,
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The main interesting property of expanders is there existence.

Theorem

For any **even number** $k \geq 4$, there exists a (k, ϵ) family of expanders Γ_n such that $|V(\Gamma_n)| = n$.

The proof of this result is a classical application of the **probabilistic method** à la Paul Erdős.

The spectral gap of a graph

Let $\Gamma = (V, E)$ be a finite regular graph of degree k.

Let $\ell^2(\Gamma) = \{x : V \to \mathbb{R}\}$, with $\|x\| = \sqrt{\sum_{v \in V} x(v)^2}$.

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The spectral gap of Γ is

$$\lambda_1(\Gamma) = \min \left\{ \frac{q_\Gamma(x)}{\|x\|^2} \,\middle|\, x \in \ell^2(\Gamma),\ \sum_{v \in V} x(v) = 0 \right\}$$

Cheeger's inequality

$$2h(\Gamma) \geq \lambda_1(\Gamma) \geq \frac{1}{2}h(\Gamma)^2.$$

Corollary

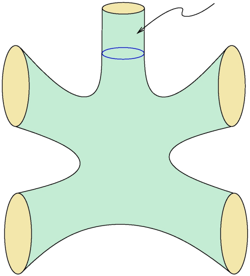
A sequence of k-regular graphs $\Gamma_n = (V_n, E_n)$ is a **family of expanders** if and only if

- Each Γ_n is a regular graph of degree k,
- $|V_n| \nearrow \infty$ as $n \to \infty$,
- $\liminf \lambda_1(\Gamma_n) > 0$.

3. Building manifolds from regular graphs

Let Γ be a finite regular graph of degree 4.

The **fundamental piece** M_0 is a compact surface such that



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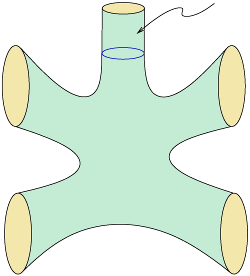

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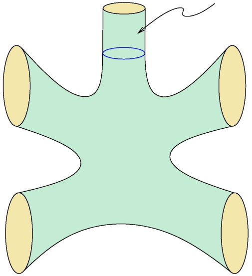

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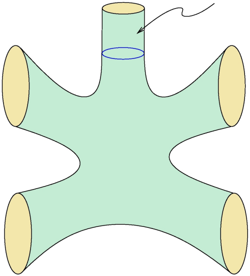

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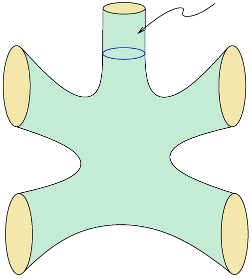

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The surface Ω_Γ is obtained by gluing copies of the fundamental piece M_0 along their boundary components $B_1, \cdots, B_4$ following the pattern of the graph Γ.

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Lemma

The genus of Ω_Γ is $1 + |V|$.

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Use x as a test function for $\lambda_1(\Gamma)$.

1. It follows from

$$\sum_v x(v) = \sum_v \int_{\Sigma_v} f = \int_{\partial\Omega} f = 0$$

that

$$\lambda_1(\Gamma) \le \frac{q_\Gamma(x).}{\|x\|^2}$$

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So that

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We will bound $q_\Gamma(x)$ and $\int_{\partial\Omega_\Gamma} \tilde{f}^2$ above in terms of

$$\int_{\Omega_\Gamma} |\nabla f|^2.$$

Lemma

There exists a constant $\mu > 0$ such that

$$\int_{\partial\Omega_\Gamma} \tilde{f}^2 \leq \mu \int_{\Omega_\Gamma} |\nabla f|^2.$$

Proof. Let μ be the first non-zero eigenvalue of the following mixed Neumann–Steklov spectral problem:

$$\Delta f = 0 \text{ in } (0,1) \times \Sigma_0,$$
$$\partial_n f = 0 \text{ on } \{1\} \times \Sigma_0, \partial_n f = \mu f \text{ on } \{0\} \times \Sigma_0.$$

It is well known that

$$\mu = \inf \left\{ \frac{\int_{(0,1)\times\Sigma_0} |\nabla f|^2}{\int_{\{0\}\times\Sigma_0} f^2} \;:\; f \in C^\infty([0,1] \times \Sigma_0), \int_{\{0\}\times\Sigma_0} f\, ds = 0 \right\}.$$

It follows that

$$\int_{\partial\Omega_\Gamma} \tilde{f}^2 \, dV_\Sigma \leq \mu^{-1} \sum_{v \in V(\Gamma)} \int_{C_v} |\nabla \tilde{f}|^2 \leq \mu^{-1} \int_\Omega |\nabla f|^2.$$

Lemma

There exists C_0 such that

$$q_\Gamma(x) = \sum_{v \sim w} \big(x(v) - x(w)\big)^2 \leq C_0 \int_{\Omega_\Gamma} |\nabla f|^2.$$

General lemma

Let A and B be two of the connected components of $\partial\Omega$. There exists $C > 0$ such that any smooth function $f \in C^\infty(\Omega)$ satisfies

$$\left| \int_A f - \int_B f \right|^2 \leq C \int_\Omega |\nabla f|^2$$

which is essentially a version of the Poincaré inequality.

To sum up $\lambda_1(\Gamma) \leq \frac{q_\Gamma(x)}{\|x\|^2}$ where

$$\|x\|^2 = \sum_v \int_{\Sigma_v} f^2 - \int_{\Sigma_v} \tilde{f}^2 = \frac{1}{\sigma_1(\Omega_\Gamma)} \|\nabla f\|^2 - \int_{\partial\Omega} \tilde{f}^2$$

Lemma

There exist constants $C, \mu > 0$ such that

$$\int_{\partial\Omega_\Gamma} \tilde{f}^2 \leq \mu \int_{\Omega_\Gamma} |\nabla f|^2, \qquad q_\Gamma(x) \leq C \int_{\Omega_\Gamma} |\nabla f|^2.$$

It follows that $\lambda_1(\Gamma) \leq \frac{C}{\sigma_1^{-1} - \mu}$, which equivalent to

$$\sigma_1 \geq \frac{\lambda_1(\Gamma)}{C + \lambda_1(\Gamma)\mu} \geq \frac{\lambda_1(\Gamma)}{C + k\mu}.$$

QED

5. Conclusion

1. This easily generalizes to higher eigenvalues.
2. The method is flexible and will adapt to other situations.
3. This naturally leads to question about the discretization of the Steklov problem in more general contexts.
   For instance, we can get large eigenvalues while keeping the number of boundary components fixed.