# Blaschke-Santaló and Mahler inequalities for the first eigenvalue of the Dirichlet Laplacian

Ilaria Fragalà, Politecnico di Milano

*ICMS Edinburgh*
*Shape Optimization and Spectral Geometry*
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Joint work with Dorin Bucur

*AIM: to launch and attack the study of new inequalities involving* polarity *on the class of convex bodies, for functionals which are* not purely geometric, *but are associated with some elliptic variational problem.*

*Polar duality on convex bodies*

For $K \in \mathscr{K}^n :=$ compact convex sets in $\mathbb{R}^n$ (with $0 \in \mathrm{int}(K)$), let

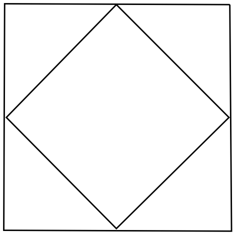
$$K^o := \Big\{ y \in \mathbb{R}^n \ : \ y \cdot x \leq 1 \quad \forall x \in K \Big\} .$$

*Examples*

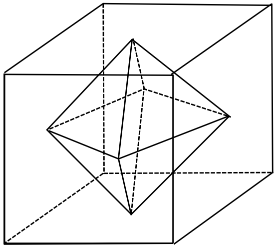
- $K = B_R(0) \ \Rightarrow \ K^o = B_{\frac{1}{R}}(0)$

- $K$ = a $n$-dim. simplex / ellipsoid/ polytope $\Rightarrow$
  $K^o$ = a $n$-dim. simplex / ellipsoid/ polytope

- $K \in \mathscr{K}^n_* := \Big\{ \text{centered bodies} \Big\} \ \Rightarrow K^o$ = unit ball for the dual norm



$(\text{Square})^o = \text{Diamond}$



$(\text{Cube})^o = \text{Octahedron}$

## The volume product

$$|K||K^o| \qquad \text{or} \qquad P(K) := \inf_{x \in \mathrm{int}(K)} |K||(K-x)^o|$$

*Remarks*:

- Invariance under invertible linear transformations
- Invariance under translations

## *Shape optimization problems*

$$\sup_{K \in \mathcal{K}^n_{(*)}} P(K) \qquad \inf_{K \in \mathcal{K}^n_{(*)}} P(K)$$

*The upper bound: Blaschke-Santaló inequality*

The supremum of $P(\cdot)$ over $\mathscr{K}^n$ is attained at the ball $B$ (and its affine images):

$$P(K) \le P(B) \qquad \forall K \in \mathscr{K}^n .$$

[Blaschke-Santaló 49]

## *The lower bound: Mahler conjecture*

In 1939 Mahler conjectured that the infimum of $P(\cdot)$ over $\mathscr{K}^n$ is attained at the **simplex** (and its affine images), and that the infimum of $P(\cdot)$ over $\mathscr{K}_*^n$ is attained at the **cube** (and its affine images) ... **STILL UNSOLVED** !

- Settled within special classes:
  - ▷ planar bodies [Mahler 39]
  - ▷ unconditional bodies [Saint-Raymond 80]
  - ▷ zonoids [Reisner 89]
  - ▷ polyhedra with a small number of vertices [Lopez-Reisner 98]
- Lower bounds for general bodies [Bourgain-Milman 87, Kuperberg 08]
- Local minima [Nazarov-Petrov-Ryabogin-Zvavitch 10, Kim-Reisner 11, Harrell-Henrot-Lamboley 14]
- Further bibliography [Schneider 14, Tao 08]

*The $\lambda_1$-product*

$$F(K) := \lambda_1(K)\lambda_1(K^o)$$

where $\lambda_1(K)$ is the first eigenvalue of the Dirichlet Laplacian on $\mathrm{int}(K)$

*Shape optimization problems*

$$\sup_{K \in \mathscr{K}^n_*} F(K) \qquad \inf_{K \in \mathscr{K}^n_*} F(K)$$

*Do these problems admit a solution?*

*What are the optimal shapes?*

*The minimization problem: Blaschke-Santaló inequality for $\lambda_1$*

**Theorem.** The ball is a minimizer for the $\lambda_1$-product:

$$\lambda_1(K)\lambda_1(K^o) \geq \lambda_1(B)\lambda_1(B^o) \qquad \forall K \in \mathcal{K}_*^n$$

*Proof:*

Based on the Faber-Krahn inequality + the Blaschke-Santaló inequality for the volume product.

## *The maximization problem: Mahler-type problem for $\lambda_1$*

*Remark*:

$$\sup_{K\in\mathcal{K}^n_*} \lambda_1(K)\lambda_1(K^o) = +\infty \qquad \text{(along a sequence of thinning domains)}$$

*Well-posedness retrieval*:

$$J(K) := \inf_{T\in GL_n} \lambda_1(T(K))\lambda_1((T(K))^o)$$

The supremum of $J$ over $\mathcal{K}^n_*$ is finite and is attained.

## *The case of axisymmetric planar bodies*

**Theorem.** The square $Q$ is a maximizer for the $\lambda_1$-product:

$$J(K) \leq J(Q) \qquad \forall K \in \mathcal{K}^2_{axi}$$

*Remark*:

$$J(Q) = \inf_T \lambda_1(T(Q))\lambda_1((T(Q))^o) = \lambda_1(Q)\lambda_1(Q^o) = \frac{\pi^4}{2}$$

**Reformulation:**

$$\forall K \in \mathcal{K}^2_{axi} \;\; \exists T \; : \; \lambda_1(T(K))\lambda_1(T(K)^o) \leq \frac{\pi^4}{2}$$

*About the proof*:

- Proof of Mahler inequality for $\lambda_1$ (axisymmetric case) $\rightsquigarrow$ proof of a reverse Faber-Krahn inequality for axisymmetric octagons.

- Prove the reverse Faber-Krahn inequality for axisymmetric octagons via a hybrid theoretical-numerical method.

## *"Reverse" inequalities*

- Reverse isoperimetric inequality

$$\inf_K \frac{|\partial K|}{|K|^{\frac{n-1}{n}}} \to \text{ball}$$

$$\sup_K \inf_T \frac{|\partial T(K)|}{|T(K)|^{\frac{n-1}{n}}} \to \text{simplex or cube}$$ [Ball 91]

- Reverse Faber-Krahn inequality

$$\inf_K \lambda_1(K)|K|^{\frac{2}{n}} \to \text{ball}$$

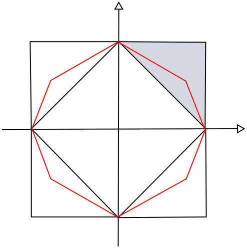
$$\sup_K \inf_T \lambda_1(T(K))|T(K)|^{\frac{2}{n}} \to ??$$

## *Reverse FK for octagons implies Mahler for $\lambda_1$ (axisymmetric case)*

Consider the family of octagons $\Omega \in \mathscr{K}^2_{axi}$ with two vertices at $(0,\ell)$ and $(\ell,0)$. Assume to know that, for any such octagon $\Omega$, it holds

$$\lambda_1(\Omega)|\Omega| \le \lambda_1(Q)|Q| = 2\pi^2 .$$

Then: $\forall K \in \mathscr{K}^2_{axi}, \; \exists\, T \; : \; \lambda_1(T(K))\lambda_1(T(K)^o) \le \frac{\pi^4}{2}$.



*Proof:*

Let $K \in \mathscr{K}^2_{axi}$. *Claim:* the body $Z := T(K)$ satisfies $\lambda_1(Z)\lambda_1(Z^o) \leq \frac{\pi^4}{2}$ if $T$ is such that

$$T(K) \cap \langle e_1 \rangle = (\pm\ell, 0) \qquad \text{and} \qquad T(K) \cap \langle e_2 \rangle = (0, \pm\ell).$$

Fix $x = (x_1, x_2) \in Z \cap (\mathbb{R}_+ \times \mathbb{R}_+)$.

By convexity $Z \supseteq \Omega^\ell_{(x_1,x_2)} :=$ the octagon with vertices in $(0,\ell)$, $(\ell,0)$, $(x_1,x_2)$.

Hence:

$$\lambda_1(Z)|\Omega^\ell_{(x_1,x_2)}| \leq \lambda_1(\Omega^\ell_{(x_1,x_2)})\,|\Omega^\ell_{(x_1,x_2)}| \leq 2\pi^2 .$$

Since $|\Omega^\ell_{(x_1,x_2)}| = 2\ell(x_1 + x_2)$, we get

$$\frac{\lambda_1(Z)(x_1 + x_2)}{\frac{\pi^2}{\ell}} \leq 1 \qquad \forall x = (x_1, x_2) \in Z .$$

Therefore,

$$\Big(\frac{\lambda_1(Z)}{\frac{\pi^2}{\ell}}, \frac{\lambda_1(Z)}{\frac{\pi^2}{\ell}}\Big) \in Z^o .$$

The same argument applied to $Z^o$ implies

$$\Big(\frac{\lambda_1(Z^o)}{\ell\pi^2}, \frac{\lambda_1(Z^o)}{\ell\pi^2}\big) \in Z .$$

*Conclusion:*

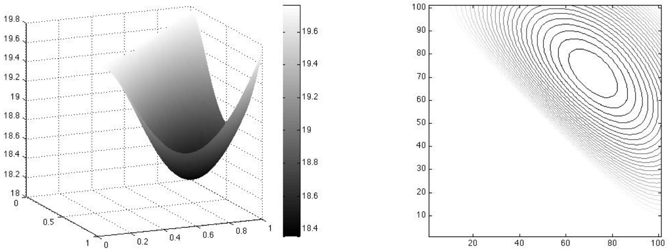
$$\frac{\lambda_1(Z)\lambda_1(Z^o)}{\pi^4} + \frac{\lambda_1(Z)\lambda_1(Z^o)}{\pi^4} \leq 1 .$$

□

*.... It remains to prove reverse FK for convex octagons !!*

$$\lambda_1(\Omega)|\Omega| \leq \lambda_1(Q)|Q| = 2\pi^2 .$$

*Numerical evidence*




Plot of the map $(x_1, x_2) \mapsto \lambda_1(\Omega_{(x_1,x_2)})|\Omega_{(x_1,x_2)}|$ and its level sets

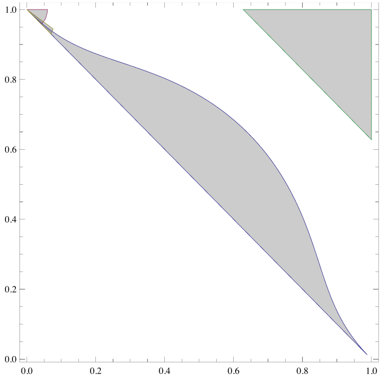
*Proof of reverse FK for convex octagons via a hybrid method*

inspired by the Polymath blog by T. Tao: http://polymathprojects.org/2013/08/09/polymath7-research-thread-5-the-hot-spots-conjecture/

- Use *theoretical arguments* to find *confidence zones*, namely computable neighborhoods of the point $P := (1,1)$ and of the segment $S := \{x_2 = 1 - x_1\}$ where the inequality holds true.

- Use a *numerical argument* (which is *mathematically rigorous*) to prove the inequality outside the confidence zones.

# Theoretical arguments: the confidence zones

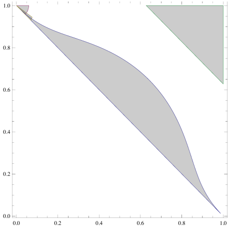


- The confidence zones are *exactly computable*:

  - ▷ $\mathscr{U}(P) := \left\{ (x_1, x_2) \in [0,1]^2 \ : \ x_1 + x_2 \geq 2 - \overline{\varepsilon} \right\}, \qquad \overline{\varepsilon} := 0.371034;$

  - ▷ $\mathscr{U}(S)$: the thickness of the confidence zone at every point of $S$ is given by the first positive zero of a fourth order polynomial;

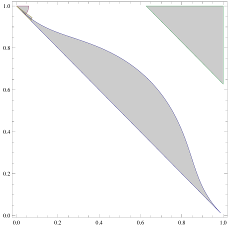
  - ▷ $\mathscr{U}(0,1)$: either $\rho \in (0, 0.06]$ and $\alpha \in \left[0, \frac{\pi}{4}\right)$, or $\rho \in (0, 0.1]$ and $\alpha \in \left[\frac{\pi}{5}, \frac{\pi}{4}\right)$.



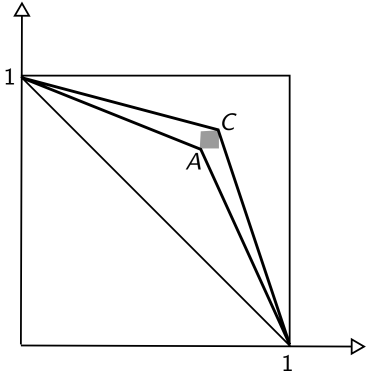
- The confidence zones are obtained by building *trial functions U* so that:

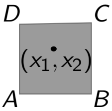
$$\lambda_1(\Omega)|\Omega| \le \frac{\int_\Omega |\nabla U|^2}{\int_\Omega |U|^2} |\Omega| \le 2\pi^2$$

  - ▷ $\mathscr{U}(P)$: use as a deformation of the square's eigenfunction for hexagons + a continuous Steiner symmetrization argument [Brock].
  - ▷ $\mathscr{U}(S)$: use two affine deformations of the diamond's eigenfunction.
  - ▷ $\mathscr{U}(0,1)$: use an affine deformation of the diamond's eigenfunction + a cut-off argument inspired by Alt-Caffarelli.



## *Numerical argument: the non-confidence zone*




By *monotonicity*:

$$\lambda_1(\Omega_{(x_1,x_2)})|\Omega_{(x_1,x_2)}| \leq \lambda_1(\Omega_A)|\Omega_C| \ \ ? < ? \ \ 2\pi^2$$

## *How to prove that* $\lambda_1(\Omega_A)|\Omega_C| < 2\pi^2$ *?*

A hand-made trial function on $\Omega_A$ is difficult to build ....

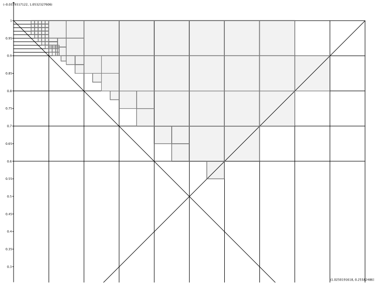
Use as test the piecewise affine function obtained by finite elements method

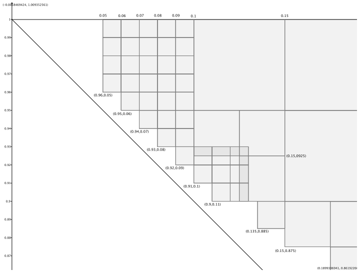
$$\lambda_1(\Omega_{(x_1,x_2)})|\Omega_{(x_1,x_2)}| \leq \lambda_1(\Omega_A)|\Omega_C| \leq \lambda_1^{num}(\Omega_A)|\Omega_C|$$

and find a covering of the non-confidence zone by squares $ABCD$ such that

$$\lambda_1^{num}(\Omega_A)|\Omega_C| < 2\pi^2$$

## *Good covering of the non-confidence region*





All the results of the computations are available at
`http://www.lama.univ-savoie.fr/~bucur/Matlab-Mahler-web/`.

## *Open problems*

- Extend the Mahler-type inequality for $\lambda_1$:

  – to the non axisymmetric case

  – to higher dimensions

  – to different functionals (such as torsional rigidity or capacity).

- Prove the reverse Faber-Krahn inequality for general convex domains (other than octagons).

THANK YOU FOR YOUR ATTENTION!