An improved energy bound for Schrödinger operators

Lorenzo Brasco

I2M – Aix-Marseille Université
lorenzo.brasco@univ-amu.fr — http://www.latp.univ-mrs.fr/~brasco/

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References

Part of the results here presented are contained in

- L. B., G. Buttazzo, Calc. Var. & PDE, **53** (2015), 977–1014

Outline

Foreword

We are in dimension $N \geq 3$ and $\mathcal{S}_N =$ "Best Sobolev constant"

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$$u \mapsto -\Delta u + V\, u$$

acting on $W_0^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; V)$

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Energy of the operator

Given $f \in W^{-1,2}(\mathbb{R}^N)$ and $V \in \mathcal{V}_{adm}$

$$\mathcal{E}_f(V) := \min \left\{ \frac{1}{2} \int |\nabla u|^2 + \frac{1}{2} \int V\,u^2 - \langle f, u \rangle \right\}$$

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$$-\Delta u_V + V\, u_V = f$$

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- we have the basic estimate
$$\|u_V\|_{W_0^{1,2}} \leq \frac{\mathcal{S}_N}{\mathcal{S}_N - \|V_-\|_{L^{N/2}}} \|f\|_{W^{-1,2}}$$

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$$\|u_V\|_{W_0^{1,2}} \leq \frac{\mathcal{S}_N}{\mathcal{S}_N - \|V_-\|_{L^{N/2}}} \|f\|_{W^{-1,2}}$$

- if $V \notin \mathcal{V}_{adm}$, it could be $\mathcal{E}_f(V) = -\infty$ (play with Sobolev extremals for a counterexample)

The problem

Let $1 < p < \infty$ and $C > 0$

$$\mathcal{V}_{int} = \{V \in \mathcal{V}_{adm} \,:\, \|V\|_{L^p} \leq C\}$$ **integrable potentials**

$$\mathcal{V}_{conf} = \left\{V \in \mathcal{V}_{adm} \,:\, \left\|\frac{1}{V}\right\|_{L^p} \leq C\right\}$$ **confining potentials**

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Question 1

Given $f \in W^{-1,2}(\mathbb{R}^N)$

$$\inf\{\mathcal{E}_f(V) \,:\, V \in \mathcal{V}_{int}\} = ? \qquad \sup\{\mathcal{E}_f(V) \,:\, V \in \mathcal{V}_{int}\} = ?$$

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Negative answers: integrable potentials

Case $p \neq N/2$

The energy is **unbounded from below** on the class $\mathcal{V}_{int}$, i.e.

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Again, we can play with Sobolev extremals to produce

$$\{V_n\}_{n\in\mathbb{N}} \subset \mathcal{V}_{int} \quad \text{and} \quad \{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,2}(\mathbb{R}^N) \quad \text{such that}$$

$$\frac{1}{2}\int |\nabla u_n|^2 + \frac{1}{2}\int V_n\, u_n^2 - \langle f, u_n\rangle \to -\infty$$

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Borderline case $p = N/2$

Existence of a minimizer in the class

$$\{V \,:\, \|V_-\|_{L^{N/2}} \leq C\} \qquad \text{for } C < \mathcal{S}_N$$

Uniqueness is not clear (for $C = \mathcal{S}_N$ and $f = 0$ uniqueness fails)

Negative answers: confining potentials

A subclass of confining potentials

Trapping potentials

$$V_\Omega = \begin{cases} 1, & \text{on } \Omega \\ +\infty, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \qquad \text{with } |\Omega| = C^p$$

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If $f \equiv 1$, this is (a suitable variant of) the **torsional rigidity**

By taking Ω_n a disjoint union of n balls of radius $C^{p/N} (\omega_N\, n)^{-1/N}$

$$\mathcal{E}_1(V_{\Omega_n}) \nearrow 0$$

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There exists a unique solution $V_0 \in \mathcal{V}_{int}$ *to problem*

$$\sup \left\{ \mathcal{E}_f(V) \,:\, V \in \mathcal{V}_{int} \right\}$$

In other words,

$$\mathcal{E}_f(V) \leq \mathcal{E}_f(V_0)$$

for every admissible V *such that* $\|V\|_{L^p} \leq C$.
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There exists a unique solution $U_0 \in \mathcal{V}_{conf}$ *to problem*

$$\inf \left\{ \mathcal{E}_f(V) \ : \ V \in \mathcal{V}_{conf} \text{ and } V \textbf{ positive} \right\}$$

In other words,

$$\mathcal{E}_f(V) \geq \mathcal{E}_f(U_0)$$

for every V **positive** *such that* $\|1/V\|_{L^p} \leq C$.
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4. the last problem has a unique solution u_0
5. if we choose U_0 so to have equality in Hölder, we get the (unique) optimal potential, i.e.
$$U_0 = \frac{1}{C}\, u_0^{-\frac{2}{p+1}}\, \|u_0\|^{2/(p+1)}_{L^{\frac{2p}{p+1}}}$$

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...is it possible to **improve** *the sharp bounds*

$$\mathcal{E}_f(V) \leq \mathcal{E}_f(V_0) \qquad \text{for} \left(\int |V|^p \right)^{1/p} \leq C$$

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by adding a **remainder term***?*

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(As in the talks by Burchard, Henrot and Fusco)

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Memento: notation

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Theorem (B.-Buttazzo)

Let $1 < p < \infty$. For every $V \in \mathcal{V}_{int}$ we have:

- *if $1 < p < 2$*

$$\mathcal{E}_f(V) \leq \mathcal{E}_f(V_0) - \sigma_1 \|V - V_0\|^2_{L^p(\Omega)}$$

- *if $p \geq 2$*

$$\mathcal{E}_f(V) \leq \mathcal{E}_f(V_0) - \sigma_1 \left\| |V|^{p-2} V - V_0^{p-1} \right\|^2_{L^{p'}(\Omega)}$$

for some (explicit) constant $\sigma_1 > 0$

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Let $1 < p < \infty$ *and* $r > N/2$. *Let* $f \in L^r(\Omega)$ *be a function such that*

$$f(x) \simeq |x|^{-\alpha} \quad \text{for } |x| \to \infty, \qquad \alpha > 1 + N/2$$

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Then for every $V \in \mathcal{V}^+_{conf}$ *we have*

$$\mathcal{E}_f(V) \geq \mathcal{E}_f(U_0) + \sigma_2 \left\| \frac{1}{V} - \frac{1}{U_0} \right\|^{\beta}_{L^p}$$

for some (explicit) constants $\sigma_2 > 0$ *and* $\beta = \beta(p) > 2$

Introduction

The results

Recipe of the proofs

Ingredients I

Reduction Lemmas

We can always reduce to prove the estimates for potentials

- saturating the constraints
- with small deficit, i.e. $\mathcal{E}_f(V) - \mathcal{E}_f(U_0) = \mathcal{E}_f(V) - \mathcal{E}_f(V_0) \simeq 0$

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Stability for convex minimizations on $W_0^{1,2}$

A functional of the form

$$J_{q,f}(u) := \frac{1}{2}\int |\nabla u|^2 + \frac{1}{2}\|u\|_{L^q}^2 - \langle f, u\rangle$$

has a unique minimizer $u_0 \in W_0^{1,2}$ and

$$J_{q,f}(u) - J_{q,f}(u_0) \geq \frac{1}{\mathcal{C}}\|u - u_0\|_{W^{1,2}}^2$$

Ingredients II

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Exponents on the right are sharp

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Sharp decay for nonautonomous Schrödinger equations

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Suppose that

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then the minimizer u_0 of

$$J_{q,f}(u) := \frac{1}{2} \int |\nabla u|^2 + \frac{1}{2} \|u\|_{L^q}^2 - \langle f, u \rangle$$

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- then we have

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and <u>both terms are positive</u>

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and both terms are positive

- by **stability for convex problems** and **quantitative Holder** (modulo some manipulations)

$$\mathcal{E}_f(V) - \mathcal{E}_f(U_0) \gtrsim \|u_V - u_0\|_{W^{1,2}}^2 + \left\| \frac{|u_V|^{\frac{2}{p+1}}}{\left(\int V\,u_V^2\right)^{\frac{1}{p+1}}} - \frac{1}{V} \right\|_{L^p}^{p+1}$$

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NEXT?

1. use optimality to write u_0 in terms of $1/U_0$ (OK, easy)
2. use "triangle inequality" to erase u_V and obtain $U_0^{-1} - V^{-1}$ (hard part)

- in order to use triangle inequality, we wish to pass from

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- to conclude we need an interpolation-type argument, based on **Clarkson inequality** and integrability of u_0 given by **sharp decay for nonautonomous equations**

Further readings

Spectral problems of this type

- *J. B. Keller*, J. Math. Phys., **2** (1961)
- *M. S. Ashbaugh, E. M. Harrell*, J. Math. Phys., **28** (1987)
- *E. A. Carlen, R. L. Frank, E. H. Lieb*, Geom. Funct. Anal., **24** (2014)

More general problems on potentials

- Antoine's Green Book (Chapter 8)
- *G. Buttazzo, A. Gerolin, B. Ruffini, B. Velichkov*, J. Éc. polytech. Math., **1** (2014)