

Eigenvalue Minimization for the Clamped Plate under Compression

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Eigenvalue Problem

$\Omega \subset \mathbb{R}^n$ bounded open.

Eigenvalue problem for clamped plate under compression $\kappa > 0$

$$\left. \begin{aligned} \Delta^2 u + \kappa \Delta u &= \lambda u && \text{in } \Omega \\ u = |\nabla u| &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (2.1)$$

We take $\kappa < \lambda_{buckling}$ where

$$\lambda_{buckling} = \inf \left\{ \frac{\int_{\Omega} |\Delta \varphi|^2 dx}{\int_{\Omega} |\nabla \varphi|^2 dx} : \varphi \in H_0^2(\Omega) \setminus \{0\} \right\} \quad (2.2)$$

whereby, the operator $\Delta^2 + \kappa \Delta$ is uniformly elliptic and self-adjoint.
First eigenvalue $\lambda = \lambda(\Omega, \kappa)$ given by the variational characterization

$$\lambda(\Omega, \kappa) := \inf \left\{ \frac{\int_{\Omega} (|\Delta \varphi|^2 - \kappa |\nabla \varphi|^2) dx}{\int_{\Omega} |\varphi|^2 dx} : \varphi \in H_0^2(\Omega) \setminus \{0\} \right\}. \quad (2.3)$$

Shape Optimization Problem

Do we have

$$\lambda(\Omega, \kappa) \geq \lambda(\Omega^*, \kappa),$$

where Ω^* is a ball of the same volume as Ω ?

History

- For the clamped plate problem ($\kappa = 0$) this is one of Rayleigh's conjectures.
 - Szegő [6, 7] showed the conjecture under the assumption that first eigenfunction u keeps same sign. Essentially by rearranging Δu in Ω .
 - But this hypothesis is not true as follows from the works of Duffin, Shaffer, Coffman.
 - For any n , Talenti [9] showed $\lambda(\Omega, \kappa) \geq c_n \lambda(\Omega^*, \kappa)$ for some constant $c_n \in (0, 1]$. c_n depends on the dimension.
 - For $n = 2$, Nadirashvili [5] shows the optimal result with $c_2 = 1$.
 - For $n = 3$ (and $n = 2$), Ashbaugh and Benguria [1] showed the optimal result with $c_3 = 1$.
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Result

Theorem (Ashbaugh, M. ; Benguria, R. ; M, R.)

For $n = 2$, we have $\lambda(\Omega, \kappa) \geq \lambda(\Omega^*, \kappa)$ for $\kappa \in [0, a]$ for some $a < \lambda_{\text{buckling}}$.

Ω^* is a ball of the same volume as Ω (and whose radius we denote by L).

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Idea

- The idea of the proof is much the same as that in Ashbaugh and Benguria [1].
- Reduce to a two-ball optimization problem [1, 2, 3, 5, 9] using a rearrangement result of Talenti [8].
- Then study the two-ball optimization problem carefully using properties of Bessel functions.
- For $n = 2$, the analysis shows us that the solution of the two-ball problem corresponds to the situation where one ball degenerates to a point for $\kappa \in [0, a]$ for some value of $a < \lambda_{buckling}$.
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Reduction to the two ball problem

We shall use the following theorem, a slight variant of a result of Talenti.

Theorem (cf. Talenti [8], Theorem 1)

Let G be a domain in R^2 , $F \in L^p(\Omega)$ for some $p > 1$ if $n = 2$. Let U be the solution of

$$\left. \begin{aligned} -\Delta U &= F & \text{in } G \\ U &= 0 & \text{on } \partial G \end{aligned} \right\} \quad (6.4)$$

and let Z be the solution of

$$\left. \begin{aligned} -\Delta Z &= F^* & \text{in } G^* \\ Z &= 0 & \text{on } \partial G^* \end{aligned} \right\} \quad (6.5)$$

where F^* is radially symmetric decreasing and equimeasurable with F on G^* . Then, if $U \geq 0$ on G , we have:

- $Z \geq U^* \geq 0$ on G^* and therefore, $\int_{G^*} |Z|^2 dx \geq \int_G |U|^2 dx$.
- $\int_{G^*} |\nabla Z|^2 dx \geq \int_G |\nabla U|^2 dx$

Note: If $n > 2$, the same result is true when $F \in L^{\frac{2n}{n+2}}(\Omega)$.

Reduction to the two ball problem

- Ω^* ball centered at the origin of radius L with volume $|\Omega|$.
- u any first eigenfunction for (2.1) in Ω . Since u may not keep the same sign, we take
 $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ and $\Omega_- = \{x \in \Omega : u(x) < 0\}$
 Ω_{\pm}^* balls with the same volume as Ω_+ and Ω_- centered at the origin, with a and b their radii.
 We have $a^2 + b^2 = L^2$.
- $f(x) = (-\Delta u)^*(x), x \in \Omega_+^*$; $g(x) = (\Delta u)^*(x), x \in \Omega_-^*$. Let v and w solve

$$\left. \begin{array}{l} -\Delta v = f \quad \text{in } \Omega_+^* \\ v = 0 \quad \text{on } \partial\Omega_+^* \end{array} \right\} \quad (6.6)$$

and

$$\left. \begin{array}{l} -\Delta w = g \quad \text{in } \Omega_-^* \\ w = 0 \quad \text{on } \partial\Omega_-^* \end{array} \right\} \quad (6.7)$$

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Reduction to the two-ball problem

- We have

$$\begin{aligned}
 \int_{\Omega} |\Delta u|^2 dx &= \int_{\Omega_+} |-\Delta u|^2 dx + \int_{\Omega_-} |\Delta u|^2 dx \\
 &= \int_{\Omega_+^*} f^2 dx + \int_{\Omega_-^*} g^2 dx \\
 &= \int_{\Omega_+^*} |\Delta v|^2 dx + \int_{\Omega_-^*} |\Delta w|^2 dx. \quad (6.8)
 \end{aligned}$$

- By Talenti's theorem, we also have

$$\int_{\Omega_+^*} |v|^2 dx \geq \int_{\Omega_+^*} |u_+|^2 dx \quad \text{and} \quad \int_{\Omega_-^*} |w|^2 dx \geq \int_{\Omega_-^*} |u_-|^2 dx \quad (6.9)$$

and

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Reduction to the two-ball problem

So, we conclude

$$\frac{\int_{\Omega} (|\Delta u|^2 - \kappa |\nabla u|^2) dx}{\int_{\Omega} |u|^2 dx} \geq \frac{\int_{\Omega_+^*} (|\Delta v|^2 - \kappa |\nabla v|^2) dx + \int_{\Omega_-^*} (|\Delta w|^2 - \kappa |\nabla w|^2) dx}{\int_{\Omega_+^*} |v|^2 dx + \int_{\Omega_-^*} |w|^2 dx}$$

This gives

$$\lambda(\Omega, \kappa) \geq J_{a,b} \tag{6.11}$$

for some a, b with $a^2 + b^2 = 1$ where

$$J_{a,b} = \min_{v,w} \frac{\int_{|x| \leq a} (|\Delta v|^2 - \kappa |\nabla v|^2) dx + \int_{|x| \leq b} (|\Delta w|^2 - \kappa |\nabla w|^2) dx}{\int_{|x| \leq a} |v|^2 dx + \int_{|x| \leq b} |w|^2 dx} \tag{6.12}$$

minimum over $v \in H^2(B_a)$, $w \in H^2(B_b)$ radial and $a \frac{\partial v}{\partial r} \Big|_{\partial B_a} = b \frac{\partial w}{\partial r} \Big|_{\partial B_b}$.

Consequently,

$$\lambda(\Omega, \kappa) \geq \min_{\Omega} \lambda(\Omega, \kappa) \geq \min_{a,b} J_{a,b}. \tag{6.13}$$

We will be done if we can show

$$\min_{a,b} J_{a,b} \geq J_{L,0} = \lambda(\Omega^*, \kappa) \tag{6.14}$$

Variational equations for the two-ball problem

Before we analyze $\min_{a,b} J_{a,b}$, let us write the variational equations at the minimum for $J_{a,b}$ for fixed a, b .

$$\left. \begin{aligned} \Delta^2 v + \kappa \Delta v &= \lambda v && \text{in } B_a \\ v &= 0 && \text{on } \partial B_a \end{aligned} \right\} \quad (6.15)$$

and

$$\left. \begin{aligned} \Delta^2 w + \kappa \Delta w &= \lambda w && \text{in } B_b \\ w &= 0 && \text{on } \partial B_b \end{aligned} \right\} \quad (6.16)$$

In addition,

$$a \frac{\partial v}{\partial r} \Big|_{\partial B_a} = b \frac{\partial w}{\partial r} \Big|_{\partial B_b} \quad (6.17)$$

and

$$\Delta v|_{\partial B_a} + \Delta w|_{\partial B_b} = 0. \quad (6.18)$$

Solution of the variational equations for the two-ball problem

For given a, b , want to obtain radial solutions v, w in the two-balls to

$$(\Delta^2 + \kappa\Delta)U = \lambda U.$$

Writing this as

$$(\Delta - \alpha^2)(\Delta + \beta^2)U = 0$$

with $(\alpha, \beta > 0)$

$$\alpha^2 = \sqrt{\lambda + \kappa^2/4} - \kappa/2, \quad \beta^2 = \sqrt{\lambda + \kappa^2/4} + \kappa/2$$

we get v, w to be of the form

$$v(r) = AJ_0(\beta r) + BI_0(\alpha r), \quad w(r) = CJ_0(\beta r) + DI_0(\alpha r). \quad (6.19)$$

$v(a) = 0 = w(b)$, $av'(a) = bw'(b)$ and $\Delta v|_{\partial B_a} + \Delta w|_{\partial B_b} = 0$ lead to a system of 4 homogeneous equations in A, B, C, D .

Solution of the variational equations for the two-ball problem







The above has a non-trivial solution iff λ is a zero of





$$F(\alpha, \beta, a) = f(\alpha, \beta, a) + f(\alpha, \beta, b), \quad (a^2 + b^2 = L^2) \quad (6.20)$$

with

$$f(\alpha, \beta, a) = a\beta \frac{J_1(\beta a)}{J_0(\beta a)} + a\alpha \frac{I_1(\alpha a)}{I_0(\alpha a)}. \quad (6.21)$$

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