Synchronization and Fluctuation Theorems for Interacting Friedman Urns

Neeraja Sahasrabudhe

Indian Institute of Technology, Bombay

Complex Networks and Emerging Applications
28th March 2016
Outline

1. Introduction
   - Background
   - Problem setup

2. Synchronization
   - $\mathcal{L}^2$-synchronization
   - Almost sure synchronization

3. Fluctuation Theorems
   - Stable Convergence
   - Stochastic Approximation

4. Other models and Future Directions
The classical Pólya urn scheme consists of an urn containing $x$ balls of one colour and $y$ balls of another colour. At time $t$, one ball is drawn randomly from the urn and its color observed; it is then replaced in the urn, along with a ball of the same color.
Initial composition of the urn: \((W_0, B_0)\). Let \(Z_t\) denote the fraction of white balls at time \(t \geq 0\).
Initial composition of the urn: \((W_0, B_0)\). Let \(Z_t\) denote the fraction of white balls at time \(t \geq 0\).

\(W_{t+1} = W_t + Y_{t+1}\), where \(Y_{t+1}\) denotes the number of white balls added to the urn at time \(t + 1\).
Initial composition of the urn: \((W_0, B_0)\). Let \(Z_t\) denote the fraction of white balls at time \(t \geq 0\).

\[ W_{t+1} = W_t + Y_{t+1}, \]

where \(Y_{t+1}\) denotes the number of white balls added to the urn at time \(t + 1\).

We have,

\[ Y_{t+1} = \begin{cases} 1 & \text{w. p. } Z_t \\ 0 & \text{w. p. } 1 - Z_t \end{cases} \]
Initial composition of the urn: \((W_0, B_0)\). Let \(Z_t\) denote the fraction of white balls at time \(t \geq 0\).

\[ W_{t+1} = W_t + Y_{t+1} , \]

where \(Y_{t+1}\) denotes the number of white balls added to the urn at time \(t + 1\).

We have,

\[ Y_{t+1} = \begin{cases} 1 \text{ w. p. } Z_t \\ 0 \text{ w. p. } 1 - Z_t \end{cases} \]

\(Z_t\) is a Martingale.
Friedman urns

Friedman urn model (proposed by Bernard Friedman in 1949) is a generalization of a Pólya urn, where the chosen ball is replaced with $\alpha$ balls of same colour and $\beta$ balls of the other colour.
If $n^{th}$ draw is a white ball. Then,

Add: $\alpha$ white and $\beta$ black balls.
If \( n^{th} \) draw is a white ball. Then,
Add: \( \alpha \) white and \( \beta \) black balls.

If \( n^{th} \) draw is a black ball. Then,
Add: \( \alpha \) black and \( \beta \) white balls.
What do we know?

- In case of Pólya urns, the fraction of balls of either colour approaches, with probability 1, a random limit that is distributed according to a beta distribution.
What do we know?

- In case of Pólya urns, the fraction of balls of either colour approaches, with probability 1, a random limit that is distributed according to a beta distribution.
- In case of Friedman urns, this limit is deterministic and is equal to $1/2$ with probability 1.
More than one urn. The reinforcement scheme depends on all urns or on a non-trivial subset of the given set of urns.
Consider $N$ urns denoted by $U(1), \ldots, U(N)$ such that at time $t = 0$ each urn contains $W_0(i) > 0$ white and $B_0(i) > 0$ black balls.
Consider $N$ urns denoted by $U(1), \ldots, U(N)$ such that at time $t = 0$ each urn contains $W_0(i) > 0$ white and $B_0(i) > 0$ black balls.

Let $N_0(i) = W_0(i) + B_0(i)$ denote the total number of balls in each urn at the beginning and $W_t(i)$ (resp. $B_t(i)$) denote the number of white balls (resp. black balls) in $U(i)$ at time $t$. 
Consider $N$ urns denoted by $U(1), \ldots, U(N)$ such that at time $t = 0$ each urn contains $W_0(i) > 0$ white and $B_0(i) > 0$ black balls.

Let $N_0(i) = W_0(i) + B_0(i)$ denote the total number of balls in each urn at the beginning and $W_t(i)$ (resp. $B_t(i)$) denote the number of white balls (resp. black balls) in $U(i)$ at time $t$.

At time $t + 1$,

$$W_{t+1}(i) = W_t(i) + Y_{t+1}(i)$$  \hspace{1cm} (1)

where $Y_{t+1}(i)$ denotes the number of white balls added to urn $U(i)$ at time $t + 1$. 
Consider \( N \) urns denoted by \( U(1), \ldots, U(N) \) such that at time \( t = 0 \) each urn contains \( W_0(i) > 0 \) white and \( B_0(i) > 0 \) black balls.

Let \( N_0(i) = W_0(i) + B_0(i) \) denote the total number of balls in each urn at the beginning and \( W_t(i) \) (resp. \( B_t(i) \)) denote the number of white balls (resp. black balls) in \( U(i) \) at time \( t \).

At time \( t + 1 \),

\[
W_{t+1}(i) = W_t(i) + Y_{t+1}(i)
\]

where \( Y_{t+1}(i) \) denotes the number of white balls added to urn \( U(i) \) at time \( t + 1 \).

We assume that \( Y_t(i) \) for \( i = 1, \ldots, N \) are conditionally independent given the past. We denote the total number of balls in each urn at time \( t \) by \( N_t(i) \).
Let \( Z_t(i) \) denote the fraction of white balls in \( U(i) \) at time \( t \) and \( Z_t \) denote the overall fraction of white balls. That is,

\[
Z_t = \frac{1}{NN_t} \sum_{i=1}^{N} W_t(i) \\
= \frac{1}{N} \sum_{i} Z_t(i)
\]
Reinforcement scheme:

We consider the following reinforcement model:

\[ P(Y_{t+1}(i) = w | \mathcal{F}_t) = \begin{cases} 
  pZ_t + (1 - p)Z_t(i) & \text{for } w = \alpha \\
  1 - pZ_t - (1 - p)Z_t(i) & \text{for } w = \beta 
\end{cases} \]

for some fixed \( p \in [0, 1] \). The parameter \( p \) is called the interaction parameter.
\begin{align*}
E[Z_{t+1}(i)|\mathcal{F}_t] &= E[W_{t+1}(i)/N_{t+1}|\mathcal{F}_t] \\
&= E\left[\frac{W_t(i) + Y_{t+1}(i)}{N_{t+1}}|\mathcal{F}_t\right] \\
&= \frac{N_t}{N_{t+1}}Z_t(i) + \frac{1}{N_{t+1}}[\alpha a_i + \beta(1 - a_i)] \\
&= \frac{N_t + (\alpha - \beta)(1 - p)}{N_{t+1}}Z_t(i) + \frac{(\alpha - \beta)p}{N_{t+1}}Z_t \\
&\quad + \frac{\beta}{N_{t+1}}
\end{align*}
\[ E[Z_{t+1}|\mathcal{F}_t] = E \left[ \frac{1}{N} \sum_i Z_{t+1}(i)|\mathcal{F}_t \right] \]
\[ = \frac{1}{N} \sum_i E[Z_{t+1}(i)|\mathcal{F}_t] \]
\[ = \frac{N_t + (\alpha - \beta)}{N_{t+1}} Z_t + \frac{\beta}{N_{t+1}} \]
Fraction of balls of a given color converges almost surely, as the time goes to infinity, to the same limit for all urns.
Theorem (S.)

Set $\rho = \frac{\alpha - \beta}{\alpha + \beta} > 0$. The following asymptotic estimates hold:

$$\text{Var}(Z_t), \text{Var}(Z_t(i)) \sim \begin{cases} 
 t^{2\rho-2} & \text{for } \rho > 1/2 \\
 t^{-1} \log t & \text{for } \rho = 1/2 \\
 t^{-1} & \text{for } \rho < 1/2 
\end{cases}$$

for every $i \in \{1, \ldots, N\}$.

$$\text{Var}(Z_t(i) - Z_t) \sim \begin{cases} 
 t^{2\rho-2\rho p-2} & \text{for } \rho > 1/[2(1-p)] \\
 t^{-1} \log t & \text{for } \rho = 1/[2(1-p)] \\
 t^{-1} & \text{for } \rho < 1/[2(1-p)] 
\end{cases}$$
For a given $i$, for $\rho < 1/2$, the $L^2$ rate of convergence of $\text{var}(Z_t)$, $\text{Var}(Z_t(i))$ and $\text{Var}(Z_t - Z_t(i))$ are the same.
For a given $i$, for $\rho < 1/2$, the $L^2$ rate of convergence of $\text{var}(Z_t)$, $\text{Var}(Z_t(i))$ and $\text{Var}(Z_t - Z_t(i))$ are the same.

In the interval $1/2 < \rho < 1/2(1 - p)$, the $\text{Var}(Z_t - Z_t(i))$ converges to zero faster than both $\text{Var}(Z_t)$ and $\text{Var}(Z_t(i))$. 
Analysis

- For a given $i$, for $\rho < 1/2$, the $L^2$ rate of convergence of $\text{var}(Z_t)$, $\text{Var}(Z_t(i))$ and $\text{Var}(Z_t - Z_t(i))$ are the same.

- In the interval $1/2 < \rho < 1/2(1 - p)$, the $\text{Var}(Z_t - Z_t(i))$ converges to zero faster than both $\text{Var}(Z_t)$ and $\text{Var}(Z_t(i))$.

- Again, for $\rho > 1/[2(1 - p)]$, $\text{Var}(Z_t(i) - Z_t)$ converges at a faster rate.
Analysis

- For a given $i$, for $\rho < 1/2$, the $L^2$ rate of convergence of $\text{var}(Z_t)$, $\text{Var}(Z_t(i))$ and $\text{Var}(Z_t - Z_t(i))$ are the same.

- In the interval $1/2 < \rho < 1/2(1 - p)$, the $\text{Var}(Z_t - Z_t(i))$ converges to zero faster than both $\text{Var}(Z_t)$ and $\text{Var}(Z_t(i))$.

- Again, for $\rho > 1/[2(1 - p)]$, $\text{Var}(Z_t(i) - Z_t)$ converges at a faster rate.

- This is a deviation from the behaviour observed in the interacting Pólya urns model.
Almost sure synchronization

Theorem (S.)

Let $Z_t$ and $Z_t(i)$ be as defined above. Then,

$$\lim_{{t \to \infty}} (Z_t - Z_t(i)) = 0 \quad \text{a.s.}$$

for every $i \in \{1, 2, \ldots, N\}$. 
Proposition

$Z_t(i)$ and $Z_t$ are quasi-martingales. That is,

$$
\sum_{t=0}^{\infty} E[|E(Z_{t+1}(i)|F_t) - Z_t(i)|] < \infty.
$$

and

$$
\sum_{t=0}^{\infty} E[|E(Z_{t+1}|F_t) - Z_t|] < \infty.
$$
In his paper titled ‘Bernard Freidman’s Urn’ (1965), David Freedman proved some important results on the fluctuations of fraction of a particular colour of balls in a Friedman urn around the limit 1/2.
Theorem (S.)

Let $\rho = \frac{\alpha - \beta}{\alpha + \beta}$. Then, we have the following:

1. For $0 < \rho < \frac{1}{2}$,

$$\sqrt{t}(Z_t - 1/2) \xrightarrow{\text{stably}} \mathcal{N} \left( 0, \frac{\rho^2}{4N(1 - 2\rho)} \right)$$

2. For $\rho = \frac{1}{2}$,

$$\frac{\sqrt{t}}{\sqrt{\log t}}(Z_t - 1/2) \xrightarrow{\text{stably}} \mathcal{N} \left( 0, \frac{\rho^2}{4N} \right)$$
Theorem (S.)

Let $\rho = \frac{\alpha - \beta}{\alpha + \beta}$. Then, we have the following:

1. For $0 < \rho < \frac{1}{2(1-p)}$,

$$\sqrt{t}(Z_t - Z_t(i)) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{(1 - \frac{1}{N}) \rho^2}{4[1 - 2\rho(1 - p)]}\right)$$

2. For $\rho = \frac{1}{2(1-p)}$,

$$\frac{\sqrt{t}}{\sqrt{\log t}}(Z_t - Z_t(i)) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \left(1 - \frac{1}{N}\right) \frac{\rho^2}{4}\right)$$
Theorem (S.)

Let \( \rho = \frac{\alpha - \beta}{\alpha + \beta} \). Then,

1. For \( \rho > \frac{1}{2} \) and \( u = 1 - \rho \),

\[
    t^u(Z_t - 1/2) \xrightarrow{\text{a.s./}L^1} \tilde{V}
\]

for some real random variable \( \tilde{V} \) such that \( P(\tilde{V} \neq 0) > 0 \).

2. For \( \rho > \frac{1}{2(1-p)} \) and \( u = 1 - (1 - p)\rho \),

\[
    t^u(Z_t - Z_t(i)) \xrightarrow{\text{a.s./}L^1} \tilde{X}
\]

for some real random variable \( \tilde{X} \) such that \( P(\tilde{X} \neq 0) > 0 \).
The notion of stable Convergence was introduced by Rényi (1963). It is, in a sense, stronger version of convergence in distribution.
Stable Convergence

Let \((\Omega, \mathcal{A}, P)\) be a probability space, and let \(S\) be a Polish space, endowed with its Borel \(\sigma\)-field. A kernel on \(S\) is a collection \(K = \{K(\omega) : \omega \in \Omega\}\) of probability measures on the Borel \(\sigma\)-field of \(S\) such that, for each bounded Borel real function \(f\) on \(S\), the map

\[
\omega \mapsto K(f)(\omega) = \int f(x) K(\omega)(dx)
\]

is \(\mathcal{A}\)-measurable.

On \((\Omega, \mathcal{A}, P)\), let \((Y_t)\) be a sequence of \(S\)-valued random variables and let \(K\) be a kernel on \(S\). Then we say that \(Y_t\) converges stably to \(K\), and we write \(Y_t \stackrel{\text{stably}}{\longrightarrow} K\), if

\[
P(Y_t \in \cdot \mid H) \stackrel{\text{weakly}}{\longrightarrow} E[K(\cdot) \mid H]
\]

for all \(H \in \mathcal{A}\) with \(P(H) > 0\).
If $Y_t \xrightarrow{\text{stably}} K$, then $Y_t$ converges in distribution to the probability distribution $E[K(\cdot)]$.

The convergence in probability of $Y_t$ to a random variable $Y$ is equivalent to the stable convergence of $Y_t$ to a special kernel, which is the Dirac kernel $K = \delta_Y$. 

A stochastic approximation scheme in $\mathbb{R}^d$ is given by

$$x(k + 1) = x(k) - a(k)h(x(k)) + a(k)M(k + 1)$$

where:

- $h : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a locally Lipschitz continuous function.
- $\{M(k)\}$ is a square-integrable martingale difference sequence with respect to increasing family of $\sigma$-fields $\mathcal{F}_k = \sigma(x(m), M(m), m \leq k)$ with
  $$E[\|M(k + 1)\|^2 | \mathcal{F}_k] \leq \infty$$
- $\{a(k)\}$ are step-sizes satisfying
  $$a(k) > 0 \ \forall k, \ \sum_k a(k) = \infty, \ \sum_k a(k)^2 < \infty.$$

The idea is that the incremental adaptation due to the slowly decreasing step-size $a(k)$ averages out the noise $\{M(k)\}$. 

**Stochastic Approximation**
Under some additional conditions, the solution of the random difference equation tracks, with probability one, the trajectory of the solution of a limiting o.d.e. given by

\[ \dot{x}(t) = -h(x(t)). \]  

(2)

as long as the iterates remain bounded.

Under additional conditions on the noise, it will converge to a (possibly random) stable equilibrium.
Theorem

Let $\delta_0 = (1/2, \ldots, 1/2)$ be an $N$-dimensional vector and let $A$ denote the $N \times N$ matrix given by

$$
\begin{pmatrix}
1/2 - \rho(1 - p) - \frac{\rho p}{N} & -\frac{\rho p}{N} & \cdots & -\frac{\rho p}{N} \\
-\frac{\rho p}{N} & 1/2 - \rho(1 - p) - \frac{\rho p}{N} & \cdots & -\frac{\rho p}{N} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\rho p}{N} & -\frac{\rho p}{N} & \cdots & 1/2 - \rho(1 - p) - \frac{\rho p}{N}
\end{pmatrix}
$$

Then,

1. For $\rho < 1/2$, $\sqrt{n}(\bar{Z}_t - \delta_0) \xrightarrow{d} \mathcal{N} \left(0, \frac{A^{-1}\rho^2}{8(1-2\rho)}\right)$

2. For $\rho = 1/2$, $\frac{\sqrt{n}}{\sqrt{\log n}}(\bar{Z}_t - \delta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma)$, for some $N \times N$ positive semi-definite matrix $\Sigma$.

3. For $\rho > 1/2$, $n^{(1-\rho)}(\bar{Z}_t - \delta_0)$ converges a.s. and in $\mathcal{L}^1$ towards a finite random variable.
Other interacting urn models

- Michel Benaïm, Itai Benjamini, Jun Chen and Yuri Lima, “A generalized Pólya’s urn with graph based interactions” (2015):

Graph-based model, with urns at each vertex and pair-wise interactions: Given a finite connected graph $G$, place a bin at each vertex. At discrete times, a ball is added to each pair of bins. In a pair of bins, one of the bins gets the ball with probability proportional to its current number of balls raised by some fixed power $\alpha > 0$.

Future directions

Reinforcement Scheme:

\[
P(Y_{t+1}(i) = w | \mathcal{F}_t) = \begin{cases} 
pZ_t + (1 - p)Z_t(i) & \text{for } w = \alpha \\
1 - pZ_t - (1 - p)Z_t(i) & \text{for } w = \beta \end{cases}
\]
Thank you!