Workshop on Extremal Kähler Metrics and Stability

Short description of the meeting

In mathematics many of the most striking problems and their solutions lie at the interface of apparently separate disciplines. This workshop brought together researchers in the separate, but related, fields of algebraic geometry, geometric analysis, symplectic geometry and differential geometry to discuss a very difficult problem at the forefront of current research. This problem is not so easy to describe in general terms. In rough terms, the extremal Kaehler metrics of the title are generalizations of metrics of constant Gauss curvature on ordinary 2-dimensional surfaces. From the point of view of analysis, they satisfy a fully nonlinear fourth- order partial differential equation, and for this type of equation no general methods are currently available. However, surfaces can also be viewed as complex algebraic curves, in other words given by polynomial equations in two variables. It is from this viewpoint that the notion of K-stability is defined.

The main theme of the workshop was to develop the conjectural relationship between the existence of extremal Kaehler metrics on the one hand, and K-stability of the underlying complex variety (higher-dimensional version of algebraic curve) on the other. This is an extremely challenging problem because of the disparate nature of the two objects being compared. While this problem remains open, the workshop gave experts in the relevant areas an opportunity to report on simplified versions of this problem where partial results have been obtained, as well as appropriate background material both from analysis and algebraic geometry. New collaborations were started during the workshop and the feedback from participants was uniformly very positive. The mix of mathematicians with different skills, focusing on a specific hard problem, has worked extremely well.

Report

This workshop focussed on two main topics:

A. Certain canonical Kahler metrics invented by Calabi called “extremal Kahler metrics”; these contain constant scalar curvature (csc) and Kahler-Einstein (KE) metrics as subclasses.
B. Notions of stability for algebraic varieties that arise in algebraic geometry when one tries to form moduli of polarised algebraic varieties using Geometric Invariant Theory.

These topics are inextricably linked through the conjectures of Yau, Tian and Donaldson that there should be a Hitchin-Kobayashi correspondence for Kahler metrics in the cohomology class c_1(L) of a polarisation L over a smooth algebraic variety: the polarised variety is stable (in a suitable sense) if and only if it admits a canonical metric (which is then unique). Yau originally made the conjecture for KE metrics and left open the right notion of stability. Tian and Donaldson developed K-stability and proposed it should be the right notion for csc Kahler metrics. At the workshop Szekelyhidi described several extensions and variants of Donaldson’s work, defining the new notions of relative, uniform, and pair K-stability. These are conjecturally equivalent to the existence

of extremal, csc, and complete csc Kahler metrics, respectively (the latter on the complement of the divisor that makes up one half of the pair). Calculations on ruled surfaces by Szekelyhidi, and by Apostolov–Calderbank–Gauduchon–Tonnesen–Friedman, also described in the talk of Calderbank, give striking evidence for these conjectures. Indeed it was examples of the latter group that suggested that K-stability alone might be too weak a notion, and motivated the development of uniform K-stability.

These conjectures were also motivated by Donaldson's deep results on metrics on toric surfaces, some of which he described in his two talks. For many participants these were the highlight of the conference. In the second he showed computer calculations of canonical metrics on toric surfaces, and on manifolds (such as a K3 surface) derived from them, displayed using state-of-the-art Q-basic graphics.

One key to Donaldson's work is to exploit the finite-dimensionality of stability to get asymptotic approximations to the infinite dimensional problem of finding canonical metrics. There were many other talks on this theme: Zelditch described work on finding finite-dimensional asymptotic approximations to geodesics in the space of Kahler metrics on a toric variety. From some viewpoints these geodesics are the key to understanding the relationship of metrics to stability, as explained in a beautiful talk by Sturm; they are approximated by the 1-parameter subgroup orbits of GIT. Chen talked about related results for arbitrary varieties and used them to get lower bounds for the Calabi functional whose extrema give these canonical metrics. This yields quick proofs that manifolds with canonical metrics must be semistable in the appropriate sense. All of these talks relied heavily on the asymptotics of the Bergman kernel, the subject of Ma's talk.

The algebraic geometry and construction of moduli side of things was represented by Alexeev, Kirwan and Viehweg, who talked about 3 completely different ways of forming moduli of varieties. Mabuchi and Ross stuck to conventional GIT, Mabuchi announcing a new and surprising result that two finite dimensional notions of stability that approach K-stability -- Chow-stability and the Hilbert-stability -- asymptotically coincide.

An obvious way to try to find canonical metrics is by minimising or controlling various energy functionals – the Calabi functional, Mabuchi functional, etc. Weinkove's talk described many of these energies, their properties and uses. Tian used the Kahler-Ricci flow in spectacular fashion to show how, for complex surfaces at least, it carries out the canonical model programme of Mori theory in a beautifully geometric way, approaching a KE metric on the canonical model.

Arezzo described a method for producing canonical metrics from canonical metrics on other manifolds by blowing up, and Le Brun lectured on beautiful applications to and from 4-manifold theory. Finally Wang gave some intriguing results and conjectures on a real, Riemannian extension of the whole theory; a surprising amount of the complex picture seems to have real analogies.

Such a focussed workshop seems to have been a great success, judging from the incredibly positive reactions of the participants. It meant that people really got to

understand almost all of all of the talks, despite the fact that in hindsight they could have been better ordered had we know who would speak on what and who would give introductory talks. It also meant that almost all of the world's experts on these topics attended. A great deal of genuine collaboration took place, often between established collaborators who took the opportunity to meet up at the conference and do some work, but also between groups who had not met before. Differential geometers learnt more about the algebro-geometric side of the theory, and vice-versa. The only pity was that while keeping the meeting small and focussed allowed us to use Maxwell's house, this meant there was little room for many graduate students or young people in related areas to attend. With hindsight we might have held the lectures in slightly larger premises.

There is much still to do in the field; in fact one would expect that the main conjecture will remain open for a decade or more. But progress is being made all the time, as we saw at the workshop, and we saw the conjectures refined and supported by new evidence even during that week. More explicit constructions will continue to be worked out, and perhaps new proofs of the conjectures in special cases (toric varieties, projective bundles over csc bases, and blow ups of manifolds with extremal metrics are the cases that are currently best understood). If Fine's work on fibered manifolds could be extended to Lefschetz fibrations then enormous families of manifolds would admit csc Kahler metrics, at least after a blow up, which would be a huge advance.

There will be more links to Sasaki–Einstein and physics (very recent developments barely touched on at the workshop) especially through csc orbifolds; Ross and Thomas took the opportunity of the workshop to spend the week starting to develop the necessary theory, which should also connect to the work of Rollin-Singer. Futaki and his collaborators have just extended Futaki invariants to the Sasaki-Einstein setting, and proved that they provide not just necessary but sufficient conditions for the existence of Sasaki-Einstein metrics in the toric case. Again, with hindsight, we should perhaps have made more time for such developments.

There were about 40 participants, from all over the world. We had 18 talks, typically 4 a day, leaving plenty of time for people to talk and work together; this proved very popular and a great deal of work was done. Below is some of the workshop feedback. This is genuinely representative, there were no negative comments on the academic programme whatsoever.

A selection of people's highlights of the workshop:

- Several excellent lectures. In particular those of younger mathematicians were quite interesting, and the general direction of research was new to me. There was enough time for discussions with (younger) mathematicians.

- The ability to hear from almost everybody who are currently researching in this area.

- Unlike most workshops and conferences which I've attended, the Kaehler Metrics

Workshop was narrowly focused an a particular problem (as opposed, say, to a general field) and, as a result, I think that the vast majority of the talks were of keen interest to the vast majority of the participants. I also very much appreciated the inclusion of some of the younger researchers (new Ph.D.'s, and graduate students) who are already making great strides in the advancement of the field.

- The subject matter treated from various points of view

- Several highlights: Hearing Donaldson's talks. Meeting a number of people in the field whom I hadn't met before. Being exposed to work I wasn't aware of. Getting together with my collaborator to get some work done. Discussing the project with others working on closely related topics

- I heard about several important breakthroughs in the field before their publication (works of Donaldson, Mabuchi, etc.). I met many people for the first time, including many promising young people whom I would not have met otherwise.

Comments on the overall academic value of the workshop:

- Absolutely wonderful - it is rare to be at a conference/workshop where virtually all talks are related to ones own research. I learned a lot.

- This was probably the most useful workshop I have ever attended. I learned a lot from the talks and from talking to the participants.

- Excellent. We were updated first-hand by the major participants in the field. Most of the active authors of the past ten years were present.

- I was impressed with the high quality of the talks and the number of new ideas which were introduced and discussed. Many speakers were able to present their work on a friendly/non-technical level, which gave me the courage to go back and try to read some of their more technical work. Also, after my talk, a I received a few comments that had a direct bearing on my work, thus providing some interesting food for thought.

- Excellent level of participants and talks, though not so much new that I hadn't already heard of. Learnt some new stuff, mainly collaborated a great deal and had opportunity to ask some questions of some experts in various fields.

- There were a lot of talks of very high level. This conference gave me a better understanding of the area, showing relations between different directions.

- This workshop was very meaningful that provided a recent overview of Kahler geometry and showed an expanse of its interests in the future. Moreover, it was valued that there was enough time for communications and discussions.

- You get to meet the active researchers of the field and getting an overview of the research field.

- The reported results were of the highest rank in the field, and in Mathematics generally. There was also sufficient time for discussion and exchange of ideas.

- Very high. It was well organized and the talks were uniformly good.

- The academic value of the workshop is as high as any that I have attended. This is a very active area, at the center of complex differential geometry and algebraic geometry. It is full of important and difficult problems, yet great progress is being steadily made.

Comments about interactions with other participants.

- I was stimulated by the lectures as well as conversations with the participants.

- I renewed a few old contacts (and started a couple of potential projects) and met several people who I knew only by name/reputation.

- I made many new contacts

- This was a unique opportunity to have comments of experts on the research I am doing. In that regard, that was very useful and many new ideas arised from discussions.

- I started collaborations with three other participants.