Report on the Workshop "Algebraic theory of Differential equations"

(Lack of) Deviation from original proposal

In our original proposal we said "The proposed Workshop is a first attempt to promote interdisciplinary research in the areas of Differential Algebra, Differential Galois Theory, their model-theoretic aspects and the theory of integrable differential and difference systems. The main objective of the Workshop is to give a strong boost to integration of these disciplines by providing a bridge across the traditional borders. A complementary objective of the Workshop is to advance research in Differential Algebra and Differential Galois Theory in the UK, bringing it to the world level."

There was no change to the objectives and the general structure was as originally proposed.

The only aspect that changed somewhat was that fewer of the model theory community were able to attend than we had hoped: however, we still had effective representation from them.

Short description of the meeting

This workshop covered Differential Algebra, Differential Galois Theory, their model-theoretic aspects and the theory of integrable differential and difference systems, areas in which, in different ways, algebraic methods are applied to differential equations. It aimed at the promotion of interdisciplinary research, so that these approaches could complement and inform one another.

Differential Galois theory works in direct analogy with usual Galois theory. Model theory has recently shown, for example, that one can always in principle compute the differential Galois group of a linear differential equation. Algebraic structures are in the heart of the theory of integrable systems. Study of Lax pairs and their reductions has led to the characterization of infinite dimensional automorphic Lie algebras, a potentially important class of algebraic structures. Non-local extensions of differential fields are required for the development of the theory of multi-dimensional integrable systems. Symmetry groups and algebras appear in several approaches, their applications together with new effective computational methods have been developed. Variational equations and isomonodromy provide one of the bridges to the differential Galois theory. All these were brought together in the workshop.

The workshop was preceded by a week-long more pedagogical school, overlapping substantially both in the areas covered and in the participants. The workshop consisted of individual talks of a more advanced character bringing participants to the frontiers of research.

Full report

This was the first meeting aimed at the promotion of interdisciplinary research in the areas of Differential Algebra, Differential Galois Theory, their model-theoretic aspects and the theory of integrable differential and difference systems. In different ways these introduce algebraic methods into the understanding and solution of differential equations, and it seemed clear that these approaches, although developed by different communities independently, should complement and inform one another.

The workshop was preceded by a week-long more pedagogical school which included the annual London Mathematical Society lecture series, given by Prof. M. Singer and titled "Introduction to the Galois Theory of Linear Differential Systems", as well as lectures on D-modules, Integrable Systems, Model Theory aspects, and Factorization of Linear Differential Equations. There was a substantial overlap between the school and workshop both in the areas covered and in the participants. The workshop consisted of individual talks of a more advanced character, rather than the several short series of introductory lectures given in the school.

There were 54 advance registrations, of which 14 were only for the school and 13 only the workshop. The 40 who attended the workshop came from 10 countries including the UK: France (7), Russia (6), USA (6), China (3), Netherlands (3), Austria (2) and one each from Finland, Germany and Spain. The Workshop was sponsored by the ICMS/EPSRC grant with a contribution from the Royal Society of Edinburgh and the Edinburgh Mathematical Society.

The structure was in general to have 5 hour-long talks per day, but one afternoon was given to 8 shorter talks, mostly by younger researchers, rather than 3 long ones. Thus there were 30 talks in all, so the majority of participants spoke. We now briefly describe these talks.

Singer's lectures on the differential Galois theory, which is based on differential fields and works in analogy with the usual Galois theory, were followed up in the workshop by the talks of Felix Ulmer and Jacques-Arthur Weil, who discussed respectively the algorithms for finding Liouvillian solutions of higher-order equations and the variational methods which enable differential Galois theory to be used in deciding whether a dynamical system is completely integrable. The link with model theory was developed in Daniel Bertrand's contrbution, which also provided links to Abelian integrals and Schanuel's conjecture relating linear independence of variables over the ratio-

nals and the transcendence degree of their exponentials. Charlotte Hardouin considered the extension to difference equations, showing how to relate Galois group structure to the (hyper-)transcendence of the solutions.

Further contributions from the logic and model theory approach were Anand Pillay's on the Grothendieck conjecture on the arithmetic generalization of linear differential equations and Alex Buium's on Arithmetic differential equations, which gave Fermat quotient analogues of the classification of 2nd order ODEs.

There were several talks on further aspects of integrability. Hieterinta provided a useful taxonomy of definitions followed by a discussion of search strategies and their outcomes, illustrated by discussion of particular cases and reference to computational tools. Vladimir Novikov discussed the perturbative symmetry approach to classification of non-linear evolution integrable equations, based on ideas of symbolic representation (due to Gelfand-Dickey) and which in effect works on successive terms in an expansion. This was followed by Wang's proof of the conjecture on the quasi-local structure of symmetries and conservation laws of multidimensional integrable equations, including the KP equation. The KP equation also appeared in Yuji Kodama's classification of multi-soliton solutions using the Sato approach and Schubert's decomposition of Grassmannians. Vladimir Sokolov showed how linear associative deformations of matrix multiplication related to integrability, and Iskander Taimanov discussed "Tame integrability".

Generalizations of methods for solution of simultaneous non-linear algebraic differential equations appeared in the talks of Alexey Ovchinnikov, who discussed characteristic sets, and Evgeny Pankratiev considered the finiteness problem for standard bases of differential ideals.

Olga Efimovskaya discussed the loop algebra decompositions and corresponding intergable equations on so(4) and Sara Lombardo, in her study of the reduction problem, has introduced infinite dimensional automorphic Lie algebras.

A morning was given to transvectants, which are useful in computing normal forms and differential invariants, which were described in talks of Jan Sanders and Chris Athorne.

Various other algorithmic approaches were discussed. Symmetries and normal forms, and effective computational methods for them, appeared in the talks of Kai Gehrs, and Moulay Barkatou, while Xiao-Shan Gao extended methods for decomposition of linear differential and difference operators. Leykin showed that his pedagogical lectures, in the school, really

provided the basis for algorithmic computuion, by giving demonstrations of computuions of Bernstein polynomials, Cech cohomology, and so on.

Georg Regensburger's talk on boundary value problems for non-linear first order ODEs was complemented by Marcus Rosenkranz's, using an operator-based approach. George Shabat gave a neat characterisation of algebraic solutions to the Painleve-VI equation in terms of Grothendiek's "dessins d'enfants". Chris Elibeck discussed addition formulae for high genus solutions of non-linear ODEs and PDEs. Finally, horizons were widened by Martin Kruskal's talk on "Surreal numbers".

That these various contributions succeeded in fulfilling our original purposes is, we believe, amply borne out by the feedback evaluation forms received. Among the many positive remarks we mention a few we especially liked: "Hard to choose between so many well-prepared talks, all trying to 'play the game'.. which was to build the bridge between dynamical systems, PDEs, differential Galois theory, number theory, model theory, symbolic computation....", "The academic value cannot be overstated...This was the first conference I've ever visited where even PhD students were integrated into the business completely. A better support for young researchers cannot be imagined.", "Very positive, the different groups seemed to interact well" and "I think that as a result of this meeting several groups of people will try to use the 'other' techniques and to understand more deeply the connections between these subjects. I think this may lead to important new directions in the future."

The practical arrangements worked very well and we echo previous organizers' tributes to the ICMS staff, even in difficult circumstances like the need to change refreshment rooms and the sudden change in airline baggage rules which happened during the week.

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