

Approximation numbers of Sobolev embeddings of radial functions

Metric Entropy and Applications in Analysis,
Learning Theory and Probability.

11-15 September 2006,
ICMS Edinburgh.

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Function spaces on \mathbb{R}^d

Sobolev spaces of fractional order, $s \in \mathbb{R}$, $1 < p < \infty$.

$$H_p^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{H_p^s} = \|\mathcal{F}^{-1}[1 + |\xi|^2]^{s/2} \mathcal{F}f\|_p < \infty\}$$

Besov spaces, $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$,

$$B_{p,q}^s(\mathbb{R}^d) = \{f \in \mathcal{S}' : \|f\|_{B_{p,q}^s} = \left(\sum_{j=0}^{\infty} 2^{sjq} \|\mathcal{F}^{-1} \varphi_j \mathcal{F}f\|_p^q \right)^{1/q} < \infty\},$$

$\psi \in C_0^\infty$, $0 \leq \psi \leq 1$, $\text{supp } \psi \subset B(0, \frac{3}{2})$, $\psi(x) = 1$ $|x| \leq 1$.

$$\varphi_0 = \psi, \varphi_1(x) = \psi\left(\frac{x}{2}\right) - \psi(x), \varphi_j(x) = \varphi_1(2^{-j+1}x)$$

Spaces of radial distributions

$SO(d)$ - the group of rotations around the origin in \mathbb{R}^d
 $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi^g(x) = \varphi(gx)$, $g \in SO(d)$, g^{-1} - inverse to g ,

$f \in \mathcal{S}'(\mathbb{R}^d)$, f^g is a distribution defined by

$$f^g(\varphi) = f(\varphi^{g^{-1}}), \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

f - radial if $f^g = f$ for any $g \in SO(d)$.

$$\begin{aligned} RH_p^s(\mathbb{R}^d) &= \{f \in H_p^s(\mathbb{R}^d) : f \text{ is radial}\}, \\ RB_{p,q}^s(\mathbb{R}^d) &= \{f \in B_{p,q}^s(\mathbb{R}^d) : f \text{ is radial}\}. \end{aligned}$$

Function spaces on symmetric spaces X

G - noncompact connected semisimple Lie group with finite center,

K - maximal compact subgroup of G ,

$X = G/K$ -associate symmetric space of dimension d ,

X - connected homogeneous Riemannian manifolds,

G - acts transitively on X as a group of isometries,

$o = eK$, then K is an isotropy group of o ,

Δ - Laplace operator on X

$\mathcal{H}_t = e^{t\Delta}$, $t \geq 0$ - positive, symmetric, semigroup of contractations in $L_p(X)$, $1 \leq p \leq \infty$, such that

$$\mathcal{H}_t 1 = 1$$

$\mathcal{S}_1(X)$ - L_1 -Schwartz space on X ,

$\mathcal{S}'_1(X)$ - space of distributions.

Let $s \in \mathbb{R}$ and $d \geq 2$

Sobolev spaces of fractional order on X , $1 < p < \infty$

$$H_p^s(X) = \left\{ f \in \mathcal{S}'_1(X) : \|f\|_{H_p^s(X)} = \left\| (I - \Delta)^{s/2} f \right\|_p < \infty \right\}.$$

Besov spaces of fractional order on \mathbf{X} ,
 $1 \leq p, q \leq \infty, k > \frac{|s|}{2}$.

$$B_{p,q}^s(\mathbf{X}) = \left\{ f \in \mathcal{S}'_1(\mathbf{X}) : \left\| f \right\|_{B_{p,q}^s(\mathbf{X})} = \left\| \mathcal{H}_{0,k} f \right\|_p + \left(\int_0^1 t^{(k-s/2)q} \left\| \frac{d^k}{dt^k} \mathcal{H}_t f \right\|_{\frac{q}{t}}^q dt \right)^{1/q} < \infty \right\},$$

$$\mathcal{H}_{0,k} f = \sum_{\ell=0}^{k-1} \frac{1}{\ell!} (-\Delta)^\ell \mathcal{H}_1 f \quad f \in \mathcal{S}'_1(\mathbf{X}).$$

Remark

$$\left(H_p^{s_0}(\mathbf{X}), H_p^{s_1}(\mathbf{X}) \right)_{\theta,q} = B_{p,q}^s(\mathbf{X}),$$

$$s = \theta s_0 + (1 - \theta) s_1, \theta \in (0, 1), s_0 \neq s_1.$$

Spaces of radial distributions

If \mathbf{K} acts transitively on sphere centred at o , then f is radial \iff is invariant w.resp. to the action of \mathbf{K}
 $f(\varphi^k) = f(\varphi), \varphi^k(x) = \varphi(k \cdot x), k \in \mathbf{K}, x \in \mathbf{X}$.

$$RH_p^s(\mathbf{X}) = \left\{ f \in H_p^s(\mathbf{X}) : f \text{ is radial} \right\},$$

$$RB_{p,q}^s(\mathbf{X}) = \left\{ f \in B_{p,q}^s(\mathbf{X}) : f \text{ is radial} \right\}.$$

Sobolev embeddings

$$(A) \quad RH_{p_0}^{s_0}(\mathbb{R}^d) \hookrightarrow RH_{p_1}^{s_1}(\mathbb{R}^d), \quad RB_{p_0, q_0}^{s_0}(\mathbb{R}^d) \hookrightarrow RB_{p_1, q_1}^{s_1}(\mathbb{R}^d)$$

$$(B) \quad RH_{p_0}^{s_0}(\mathbf{X}) \hookrightarrow RH_{p_1}^{s_1}(\mathbf{X}), \quad RB_{p_0, q_0}^{s_0}(\mathbf{X}) \hookrightarrow RB_{p_1, q_1}^{s_1}(\mathbf{X})$$

Continuity of Sobolev embedd. of radial functions

If $p_0 \leq p_1$ and $s_0 - \frac{d}{p_0} \geq s_1 - \frac{d}{p_1} \implies$

the embeddings (A) and (B) are bounded

(if $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$ then $q_0 \leq q_1$ for Besov spaces).

Compactness of Sobolev embedd. of radial functions

Let $d \geq 2$. The embeddings (A) and (B) are compact

$\iff p_0 < p_1$ and $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$.

Approximation numbers

X, Y - complex Banach spaces,

$T : X \rightarrow Y$ - bounded linear operator, $k \in \mathbb{N}$,

$$a_k(T) := \inf\{\|T - A\| : A \in L(X, Y), \text{rank}(A) < k\},$$
$$\text{rank}(T) = \dim T(X).$$

Properties of approximation numbers:

- $a_k(T)$ - decreasing sequence $a_1(T) = \|T\|$
- $a_k \rightarrow 0$, for $k \rightarrow \infty$, $\Rightarrow T$ is compact,
- $a_{n+k-1}(T_1 + T_2) \leq a_n(T_1) + a_k(T_2)$,
- $a_{n+k-1}(T_1 \circ T_2) \leq a_n(T_1)a_k(T_2)$,
- $a_k(T) = 0 \iff \text{rank}(T) < k$.

Asymptotic behaviour of a_k in cases: \mathbb{R}^d and X

Theorem 1 Let $1 < p_0 < p_1 \leq \infty$, $(p_0, p_1) \neq (1, \infty)$, $d \geq 2$, $\frac{1}{p} = \frac{1}{p_0} - \frac{1}{p_1}$ and $t = \min\{p'_0, p_1\}$, $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$, then

$$a_k(\text{id}_{\mathbf{A}}) \sim k^{-\beta(\mathbf{A})}, \quad a_k(\text{id}_{\mathbf{B}}) \sim k^{-\beta(\mathbf{B})}$$

where $\beta(\mathbf{A}) =$

$$\begin{cases} \frac{d-1}{p} & \text{if } 1 \leq p_0 < p_1 \leq 2 \text{ or } 2 \leq p_0 < p_1 \leq \infty, \\ \frac{d-1}{p} + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, \text{ and } \frac{d-1}{p} \geq \frac{1}{t}, \\ \frac{d-1}{p} \frac{t}{2} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty \text{ and } \frac{d-1}{p} < \frac{1}{t}. \end{cases}$$

and $\beta(\mathbf{B}) =$

$$\begin{cases} s - \frac{1}{p} & \text{if } 1 \leq p_0 < p_1 \leq 2 \text{ or } 2 \leq p_0 < p_1 \leq \infty, \\ s - \frac{1}{p} + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty \text{ and } s - \frac{1}{p} \geq \frac{1}{t}, \\ \left(s - \frac{1}{p}\right) \frac{t}{2} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty \text{ and } s - \frac{1}{p} < \frac{1}{t}, \end{cases}$$

where $s = s_0 - s_1$.

Proof in case of \mathbb{R}^d

Upper estimate on \mathbb{R}^d - reduction to sequence spaces.

Weighted seq. spaces: $1 \leq p, q \leq \infty$, $\delta \geq 0$,
 $w : \mathbb{N}_0 \rightarrow \mathbb{R}_+$.

$$\ell_q(2^{j\delta} \ell_p(w)) = \left\{ (s_{j,k})_{j,k} : \|s_{j,k}\|_{\ell_q(2^{j\delta} \ell_p(w))} = \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} 2^{j\delta p} w(k) |s_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}. (**)$$

(with the usual modification if $p = \infty$ or $q = \infty$).

Lemma 1 *Let $p_0 < p_1$ and $1 \leq q_0, q_1 \leq \infty, s_0 > 0$, $\delta = s_0 - s_1 - d(\frac{1}{p_0} - \frac{1}{p_1})$ and $\delta > 0$, then there are bounded operators S and T such that*

$$\begin{array}{ccc} RB_{p_0, q_0}^{s_0}(\mathbb{R}^d) & \xrightarrow{S} & \ell_{q_0}(2^{j\delta} \ell_{p_0}(w_{d-1})) \\ \text{id} \downarrow & & \downarrow \text{Id} \\ RB_{p_1, q_1}^{s_1}(\mathbb{R}^d) & \xleftarrow{T} & \ell_{q_1}(\ell_{p_1}(w_{d-1})). \end{array}$$

is commutative.

$$w_\alpha(k) = (1 + k)^\alpha \quad (\star).$$

Let $1 \leq p_0 < p_1 \leq \infty$, $(p_0, p_1) \neq (1, \infty)$, $\frac{1}{p} = \frac{1}{p_0} - \frac{1}{p_1}$, $\alpha > 0$, $t = \min\{p_0', p_1\}$.

Proposition 1 Let D_α be a diagonal operator generated by the sequence $\sigma_k = k^{-\alpha}$, then

$$a_k \left(D_\alpha : l_{p_0} \rightarrow l_{p_1} \right) \sim C k^{-\beta}$$

$$\beta = \begin{cases} \alpha & \text{if } 1 \leq p_0 < p_1 \leq 2 \text{ or } 2 \leq p_0 < p_1 \leq \infty, \\ \alpha + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, \alpha > \frac{1}{t}, \\ \alpha \frac{t}{2} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, \alpha \leq \frac{1}{t}. \end{cases}$$

Proposition 2 Let $1 \leq q_0, q_1 \leq \infty$ and $\delta > 0$, then

$$a_k \left(\text{Id} : l_{q_0}(2^{j\delta} l_{p_0}(w_\alpha)) \rightarrow l_{q_1}(l_{p_1}, (w_\alpha)) \right) \sim k^{-\beta}.$$

$$\beta = \begin{cases} \frac{\alpha}{p} & \text{if } 1 \leq p_0 < p_1 \leq 2 \text{ or } 2 \leq p_0 < p_1 \leq \infty, \\ \frac{\alpha}{p} + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, \frac{\alpha}{p} \geq \frac{1}{t}, \\ \frac{\alpha t}{p 2} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, \frac{\alpha}{p} < \frac{1}{t}. \end{cases}$$

Lower estimates on \mathbb{R}^d - by trace operator to the weighted function spaces.

For $s_0 > s_1 > 0$ the following diagram is commutative

$$\begin{array}{ccc}
 B_{p_0, q_0}^{s_0}(\mathbb{R}_+, w_{\frac{(d-1)}{p_0}}, [1, \infty)) & \xrightarrow{\text{ext}} & RB_{p_0, q_0}^{s_0}(\mathbb{R}^d, [1, \infty)) \\
 \text{Id}_2 \downarrow & & \downarrow \text{Id} \\
 B_{p_1, q_1}^{s_1}(\mathbb{R}_+, w_{\frac{(d-1)}{p_1}}, [1, \infty)) & \xleftarrow{\text{tr}^*} & RB_{p_1, q_1}^{s_1}(\mathbb{R}^d, [1, \infty)),
 \end{array}$$

Theorem 2 Let $1 \leq p_0 \leq p_1 \leq \infty$, $(p_0, p_1) \neq (1, \infty)$, $1 \leq q_0, q_1 \leq \infty$, $-\infty < s_1 < s_0 < \infty$ and $\alpha > 0$ and $\alpha \neq s_0 - s_1 - \left(\frac{1}{p_0} - \frac{1}{p_1}\right) =: \tilde{\delta}$, then

$$a_k(B_{p_0, q_0}^{s_0}(\mathbb{R}, w_\alpha) \hookrightarrow B_{p_1, q_1}^{s_1}(\mathbb{R})) \sim k^{-\varkappa},$$

where

$$\varkappa = \begin{cases} \min\{\alpha, \tilde{\delta}\} & \text{if } 1 \leq p_0 < p_1 \leq 2 \\ & \text{or } 2 \leq p_0 < p_1 \leq \infty, \\ \min\{\alpha, \tilde{\delta}\} + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, \\ & \text{and } \min\{\alpha, \tilde{\delta}\} > \frac{d}{t}, \\ \min\{\alpha, \tilde{\delta}\} \cdot \frac{t}{2} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty, \\ & \text{and } \min\{\alpha, \tilde{\delta}\} \leq \frac{d}{t}. \end{cases}$$

$$\alpha = (d-1)\left(\frac{1}{p_0} - \frac{1}{p_1}\right), \quad \min\{\alpha, \tilde{\delta}\} = \alpha.$$

Proof in case of X

Local part (near origin o) - as on Euclidean ball.

$$\ell_q(2^{j\delta} \ell_p^{\gamma 2^j}(w)) = \left\{ (s_{j,k})_{j,k} \in \ell_q(2^{j\delta} \ell_p(w)) : s_{j,k} = 0 \text{ if } k > \gamma 2^j \right\}.$$

Lemma 2 *Let $\gamma > 0$, then*

$$a_k(\text{id} : \ell_{q_0}(2^{j\delta} \ell_{p_0}^{\gamma 2^j}(w_\alpha)) \rightarrow \ell_{q_1}(\ell_{p_1}^{\gamma 2^j}(w_\alpha))) \sim k^{-\beta},$$

$$\beta = \begin{cases} \delta + \frac{\alpha}{p} & \text{for } 1 \leq p_0 < p_1 \leq 2 \\ & \text{or } 2 \leq p_0 < p_1 \leq \infty, \\ \delta + \frac{\alpha}{p} + \frac{1}{2} - \frac{1}{t} & \text{for } 1 \leq p_0 < 2 < p_1 \leq \infty \\ & \text{and } \delta + \frac{\alpha}{p} \geq \frac{1}{t}, \\ (\delta + \frac{\alpha}{p}) \frac{t}{2} & \text{for } \leq p_0 < 2 < p_1 \leq \infty \\ & \text{and } \delta + \frac{\alpha}{p} < \frac{1}{t}, \end{cases}$$

$$\alpha = d - 1, \quad \delta = s_0 - s_1 - d \left(\frac{1}{p_0} - \frac{1}{p_1} \right), \quad w_\alpha = (\star).$$

Global part (out of origine o) - by trace operator on geodesic rays γ to weighted function spaces.

$$\begin{aligned} \text{tr} & : RB_{p,q}^s(\mathbf{X} \setminus B(o, \tau)) \mapsto B_{p,q}^s(\gamma[\tau, \infty), v_p), \\ \text{ext} & : B_{p,q}^s(\gamma[\tau; \infty), v_p) \mapsto RB_{p,q}^s(\mathbf{X} \setminus B(o, \tau)). \end{aligned}$$

$\tau > 0$

$$\begin{aligned} a_k \left(B_{p_0, q_0}^{s_0}(\mathbb{R}, v_{p_0}) \hookrightarrow B_{p_1, q_1}^{s_1}(\mathbb{R}, v_{p_1}) \right) & \sim \\ a_k \left(\ell_{q_0} \left(2^{j\delta} \ell_{p_0}(v_\alpha) \right) \hookrightarrow \ell_{q_1}(\ell_{p_1}) \right) & \end{aligned}$$

$$v_p(t) = \exp \frac{\vartheta|t|}{p} \text{ if } |t| \geq 1$$

Lemma 3 *Let $\alpha, \delta > 0$, then*

$$a_k \left(\text{id} : \ell_{q_0} \left(2^{j\delta} \ell_{p_0}(v_\alpha) \right) \rightarrow \ell_{q_1}(\ell_{p_1}) \right) \sim k^{-\varkappa},$$

$$\varkappa = \begin{cases} \delta & \text{if } 1 \leq p_0 < p_1 \leq 2 \\ & \text{or } 2 \leq p_0 < p_1 \leq \infty, \\ \delta + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty \\ & \text{and } \delta > \frac{1}{t}, \\ \frac{\delta t}{2} & \text{if } 1 \leq p_0 < 2 < p_1 \leq \infty \\ & \text{and } \delta \leq \frac{1}{t}. \end{cases}$$

$$v_\alpha(j, k) = \exp(\alpha 2^{-j} k) \quad \alpha = \frac{\vartheta}{p}.$$

Entropy numbers of id_A and id_B

Theorem 3 *Let $1 < p_0 < p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$, $s_0 - s_1 - d \left(\frac{1}{p_0} - \frac{1}{p_1} \right) > 0$.*

$$e_k(\text{id}_A) \sim k^{-d \left(\frac{1}{p_0} - \frac{1}{p_1} \right)}$$

$$e_k(\text{id}_B) \sim k^{-(s_0 - s_1)}$$

References

- 1) T.Kühn, H.-G.Leopold, W.Sickel, L.Skrzypczak, *Entropy numbers of Sobolev embeddings of radial Besov spaces*. J. Approx. Theory **121**(2003),
- 2) T.Kühn, H.-G.Leopold, W.Sickel, L.Skrzypczak, *Entropy numbers of embeddings of weighted Besov spaces*. Jeanaer Schriften zur Math. und Inf. **13**(2003),
- 3)W.Sickel, L.Skrzypczak, *Radial subspaces of Besov and Triebel-Lizorkin spaces: extended Strauss lemma and compactness of embeddings*. J. Fourier Anal. Appl. **6** (2000),
- 4)L.Skrzypczak, *Heat extensions, optimal atomic decompositions and Sobolev embeddings in presence of symmetries on manifolds*. Math. Z. **243**(2003),
- 5)L.Skrzypczak, *On approximation numbers of Sobolev embeddings of weighted function spaces*, J. Approx. Theory **136**(1) (2005),
- 6)L.Skrzypczak, *Rotation invariant subspaces of Besov and Triebel-Lizorkin spaces: compactness of embeddings, smoothness and decay properties*. Revista Mat. Iberoamericana **18**(2002),
- 7)L. Skrzypczak, B.T, *Approximation numbers of Sobolev embeddings between radial Besov and Sobolev spaces*, Comment. Math. Tomus Specialis in Honorem Iuliani Musielak (2004),
- 8)L. Skrzypczak, B.T, *Approximation numbers of Sobolev embeddings of radial functions on isotropic manifolds*, JFSA.