

# Approximation numbers in anisotropic function spaces

Erika Tamási

Friedrich-Schiller-University Jena, Germany

Edinburgh  
September 2006

# Outline

1. Definitions
  - 1.1 Anisotropic Besov spaces
  - 1.2 Traces and approximation numbers
  - 1.3 Anisotropic  $d$ -sets in  $\mathbb{R}^n$
2. Results
  - 2.1 Main result
  - 2.2 Proof of the main result
3. Applications

## Anisotropic dyadic resolution of unity

Let  $\varphi_0 \in \mathcal{S}$  be such that

$$\varphi_0(\mathbf{x}) = 1 \quad \text{if} \quad |\mathbf{x}|_a \leq 1 \quad \text{and} \quad \text{supp } \varphi_0 \subset \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}|_a \leq 2\},$$

and for each  $j \in \mathbb{N}$  let

$$\varphi_j^a(\mathbf{x}) := \varphi_0(2^{-ja}\mathbf{x}) - \varphi_0(2^{-(j+1)a}\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

## Anisotropic dyadic resolution of unity

Let  $\varphi_0 \in \mathcal{S}$  be such that

$$\varphi_0(\mathbf{x}) = 1 \quad \text{if} \quad |\mathbf{x}|_a \leq 1 \quad \text{and} \quad \text{supp } \varphi_0 \subset \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}|_a \leq 2\},$$

and for each  $j \in \mathbb{N}$  let

$$\varphi_j^a(\mathbf{x}) := \varphi_0(2^{-ja}\mathbf{x}) - \varphi_0(2^{-(j+1)a}\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

## Anisotropic Besov spaces

Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ ,  $s \in \mathbb{R}$ , then

$$B_{pq}^{s,a}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \|f\|_{B_{pq}^{s,a}(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j^a \hat{f})^\vee\|_{L_p(\mathbb{R}^n)}^q \right)^{1/q} < \infty \right\}$$

## Anisotropic dyadic resolution of unity

Let  $\varphi_0 \in \mathcal{S}$  be such that

$$\varphi_0(\mathbf{x}) = 1 \quad \text{if} \quad |\mathbf{x}|_a \leq 1 \quad \text{and} \quad \text{supp } \varphi_0 \subset \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}|_a \leq 2\},$$

and for each  $j \in \mathbb{N}$  let

$$\varphi_j^a(\mathbf{x}) := \varphi_0(2^{-ja}\mathbf{x}) - \varphi_0(2^{-(j+1)a}\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

## Anisotropic Besov spaces

Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ ,  $s \in \mathbb{R}$ , then

$$B_{pq}^{s,a}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \|f\|_{B_{pq}^{s,a}(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j^a \hat{f})^\vee\|_{L_p(\mathbb{R}^n)}^q \right)^{1/q} < \infty \right\}$$

## Traces

Let  $\Gamma$  be a closed set in  $\mathbb{R}^n$  with  $|\Gamma| = 0$ ,  $\Gamma \neq \emptyset$

## Traces

Let  $\Gamma$  be a closed set in  $\mathbb{R}^n$  with  $|\Gamma| = 0$ ,  $\Gamma \neq \emptyset$

- for any  $\varphi \in \mathcal{S}$  then  $tr_{\Gamma}\varphi = \varphi|_{\Gamma}$  makes sense pointwise

## Traces

Let  $\Gamma$  be a closed set in  $\mathbb{R}^n$  with  $|\Gamma| = 0$ ,  $\Gamma \neq \emptyset$

- for any  $\varphi \in \mathcal{S}$  then  $tr_{\Gamma}\varphi = \varphi|_{\Gamma}$  makes sense pointwise
- assume that for some  $s > 0$ ,  $0 < p, q < \infty$  there exists a constant  $c > 0$  such that

$$\|tr_{\Gamma}\varphi|_{L_p(\Gamma)}\| \leq c\|\varphi|_{B_{pq}^{s,a}(\mathbb{R}^n)}\|, \quad \varphi \in \mathcal{S}$$

## Traces

Let  $\Gamma$  be a closed set in  $\mathbb{R}^n$  with  $|\Gamma| = 0$ ,  $\Gamma \neq \emptyset$

- for any  $\varphi \in \mathcal{S}$  then  $tr_{\Gamma}\varphi = \varphi|_{\Gamma}$  makes sense pointwise
- assume that for some  $s > 0$ ,  $0 < p, q < \infty$  there exists a constant  $c > 0$  such that

$$\|tr_{\Gamma}\varphi\|_{L_p(\Gamma)} \leq c\|\varphi\|_{B_{pq}^{s,a}(\mathbb{R}^n)}, \quad \varphi \in \mathcal{S}$$

- $\mathcal{S}$  is dense in  $B_{pq}^{s,a}(\mathbb{R}^n)$  and the inequality can be extended to any  $f \in B_{pq}^{s,a}(\mathbb{R}^n)$  and the resulting function is denoted by  $tr_{\Gamma}f$

## Traces

Let  $\Gamma$  be a closed set in  $\mathbb{R}^n$  with  $|\Gamma| = 0$ ,  $\Gamma \neq \emptyset$

- for any  $\varphi \in \mathcal{S}$  then  $tr_{\Gamma}\varphi = \varphi|_{\Gamma}$  makes sense pointwise
- assume that for some  $s > 0$ ,  $0 < p, q < \infty$  there exists a constant  $c > 0$  such that

$$\|tr_{\Gamma}\varphi\|_{L_p(\Gamma)} \leq c\|\varphi\|_{B_{pq}^{s,a}(\mathbb{R}^n)}, \quad \varphi \in \mathcal{S}$$

- $\mathcal{S}$  is dense in  $B_{pq}^{s,a}(\mathbb{R}^n)$  and the inequality can be extended to any  $f \in B_{pq}^{s,a}(\mathbb{R}^n)$  and the resulting function is denoted by  $tr_{\Gamma}f$

## Approximation numbers

Let  $A$  and  $B$  be two Banach spaces,  $T \in L(A, B)$

$$a_k(T) = \inf\{\|T - L\| : L \in L(A, B), \text{rank}L < k\}$$

## Anisotropic $d$ -sets in $\mathbb{R}^n$

Let  $0 < d < n$  and  $\mu$  a positive Radon measure in  $\mathbb{R}^n$  with  $\Gamma = \text{supp } \mu$ ,  $0 < \mu(\mathbb{R}^n) < \infty$ ,  $|\Gamma| = 0$ .  $\Gamma \subset \mathbb{R}^n$  is called an **anisotropic  $d$ -set** if

$$\mu(B^a(\gamma, r)) \sim r^d, \quad 0 < r < 1,$$

where  $B^a(\gamma, r) = \{y \in \mathbb{R}^n : |y - \gamma|_a \leq r\}$  and  $\gamma \in \Gamma$ .

## Anisotropic $d$ -sets in $\mathbb{R}^n$

Let  $0 < d < n$  and  $\mu$  a positive Radon measure in  $\mathbb{R}^n$  with  $\Gamma = \text{supp } \mu$ ,  $0 < \mu(\mathbb{R}^n) < \infty$ ,  $|\Gamma| = 0$ .  $\Gamma \subset \mathbb{R}^n$  is called an **anisotropic  $d$ -set** if

$$\mu(B^a(\gamma, r)) \sim r^d, \quad 0 < r < 1,$$

where  $B^a(\gamma, r) = \{y \in \mathbb{R}^n : |y - \gamma|_a \leq r\}$  and  $\gamma \in \Gamma$ .

### Proposition:

For every  $0 < d < n$  there exists an anisotropic  $d$ -set.

# Main result

## Theorem

Let the anisotropic  $d$ -set  $\Gamma$  and the Radon measure  $\mu$  be as in the definition, and let

$$0 < d < n, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{n}{p} \geq s > \frac{n-d}{p}.$$

Let  $a_k = a_k(\text{tr}_\Gamma)$  be the approximation numbers of the compact trace operator  $\text{tr}_\Gamma$ . Then there exist numbers  $c, c' > 0$  so that for all  $k \in \mathbb{N}$

$$ck^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}} \leq a_k(\text{tr}_\Gamma : B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)) \leq c'k^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}}.$$

# Recall

## Theorem 2.1, D.H., E.T.

Let  $0 < p < \infty$ ,  $s > \sigma_p$ ,  $\varrho \geq 0$ , and  $a$  an anisotropy. Then

$$f \in B_{pp}^{s,a}(\mathbb{R}^n) \iff f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{\beta} k^{\beta} (2^{ja} x - m), \quad x \in \mathbb{R}^n,$$

with  $\|\lambda|b_p^{s,\varrho}\| < \infty$ , absolute convergence being in  $L_{\max(1,p)}$ . Furthermore,

$$\|f|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim \inf \|\lambda|b_p^{s,\varrho}\|.$$

In addition, any  $f \in B_{pp}^{s,a}(\mathbb{R}^n)$  can be optimally represented by

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{\beta}(f) k^{\beta} (2^{ja} x - m), \quad \text{with} \quad \|f|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim \|\lambda(f)|b_p^{s,\varrho}\|.$$

▶ Proof ideas, right-hand side:

▶ Proof ideas, right-hand side:

$$\bullet \operatorname{tr}_r^\beta f = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta(f) k_{jm}^\beta \quad \text{and} \quad \operatorname{tr}_r^{\beta, J} f = \sum_{j \leq J} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta(f) k_{jm}^\beta$$

▶ Proof ideas, right-hand side:

$$\bullet \operatorname{tr}_\Gamma^\beta f = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta(f) k_{jm}^\beta \quad \text{and} \quad \operatorname{tr}_\Gamma^{\beta, J} f = \sum_{j \leq J} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^\beta(f) k_{jm}^\beta$$

$m \in \mathbb{Z}^n$  where  $Q_{jm}^a \cap \Gamma \neq \emptyset$

▶ Proof ideas, right-hand side:

- $tr_{\Gamma}^{\beta} f = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}$  and  $tr_{\Gamma}^{\beta, J} f = \sum_{j \leq J} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}$
- $\|(tr_{\Gamma}^{\beta} - tr_{\Gamma}^{\beta, J})f\|_{L_p(\Gamma)} \leq c 2^{-\delta a \beta} 2^{J(\frac{n}{p} - s)} 2^{-J \frac{d}{p}}$

▶ Proof ideas, right-hand side:

- $tr_{\Gamma}^{\beta} f = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n}^{\Gamma} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}$  and  $tr_{\Gamma}^{\beta, J} f = \sum_{j \leq J} \sum_{m \in \mathbb{Z}^n}^{\Gamma} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}$

- $\|(tr_{\Gamma}^{\beta} - tr_{\Gamma}^{\beta, J})f\|_{L_p(\Gamma)} \leq c 2^{-\delta a \beta} 2^{J(\frac{n}{p} - s)} 2^{-J \frac{d}{p}}$

- we put  $L = tr_{\Gamma}^{\beta, J}$ ,  $T = tr_{\Gamma}^{\beta}$ , and note that for  $j \in \mathbb{N}_0$

$$\text{rank} \left( \sum_{m \in \mathbb{Z}^n}^{\Gamma} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta} \right) \leq c 2^{jd} \Rightarrow \text{rank}(tr_{\Gamma}^{\beta, J}) \leq c \sum_{j \leq J} 2^{jd} \leq c' 2^{Jd}$$

► Proof ideas, right-hand side:

- $tr_{\Gamma}^{\beta} f = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \sum_{\Gamma} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}$  and  $tr_{\Gamma}^{\beta, J} f = \sum_{j \leq J} \sum_{m \in \mathbb{Z}^n} \sum_{\Gamma} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}$

- $\|(tr_{\Gamma}^{\beta} - tr_{\Gamma}^{\beta, J})f\|_{L_p(\Gamma)} \leq c 2^{-\delta a \beta} 2^{J(\frac{n}{p}-s)} 2^{-J\frac{d}{p}}$

- we put  $L = tr_{\Gamma}^{\beta, J}$ ,  $T = tr_{\Gamma}^{\beta}$ , and note that for  $j \in \mathbb{N}_0$

$$\text{rank}\left(\sum_{m \in \mathbb{Z}^n} \sum_{\Gamma} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}\right) \leq c 2^{jd} \Rightarrow \text{rank}(tr_{\Gamma}^{\beta, J}) \leq c \sum_{j \leq J} 2^{jd} \leq c' 2^{Jd}$$

- $a_{c 2^{Jd}}(tr_{\Gamma}^{\beta}) \leq c' 2^{-\delta a \beta} 2^{J(\frac{n}{p}-s)} 2^{-J\frac{d}{p}} \Rightarrow a_{ck}(tr_{\Gamma}^{\beta}) \leq c 2^{-\delta a \beta} 2^{J_k(\frac{n}{p}-s)} k^{-\frac{1}{p}}$   
 where  $2^{J_k d} \sim k$

► Proof ideas, right-hand side:

- $tr_{\Gamma}^{\beta} f = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}$  and  $tr_{\Gamma}^{\beta, J} f = \sum_{j \leq J} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}$

- $\|(tr_{\Gamma}^{\beta} - tr_{\Gamma}^{\beta, J})f\|_{L_p(\Gamma)} \leq c 2^{-\delta a \beta} 2^{J(\frac{n}{p}-s)} 2^{-J\frac{d}{p}}$

- we put  $L = tr_{\Gamma}^{\beta, J}$ ,  $T = tr_{\Gamma}^{\beta}$ , and note that for  $j \in \mathbb{N}_0$

$$\text{rank}\left(\sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{\beta}(f) k_{jm}^{\beta}\right) \leq c 2^{jd} \Rightarrow \text{rank}(tr_{\Gamma}^{\beta, J}) \leq c \sum_{j \leq J} 2^{jd} \leq c' 2^{Jd}$$

- $a_{c 2^{Jd}}(tr_{\Gamma}^{\beta}) \leq c' 2^{-\delta a \beta} 2^{J(\frac{n}{p}-s)} 2^{-J\frac{d}{p}} \Rightarrow a_{ck}(tr_{\Gamma}^{\beta}) \leq c 2^{-\delta a \beta} 2^{J_k(\frac{n}{p}-s)} k^{-\frac{1}{p}}$   
 where  $2^{J_k d} \sim k$

$$\Rightarrow a_{ck}(tr_{\Gamma}^{\beta}) \leq c''' k^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}}$$

▶ Proof ideas, left-hand side:

▶ Proof ideas, left-hand side:

• let

$$f_j^a(\mathbf{x}) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(\mathbf{x} - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad \mathbf{x} \in \mathbb{R}^n$$

▶ Proof ideas, left-hand side:

• let

$$f_j^a(\mathbf{x}) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(\mathbf{x} - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad \mathbf{x} \in \mathbb{R}^n$$

$M_j \sim 2^{jd}$

▶ Proof ideas, left-hand side:

• let

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(x - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n$$
$$\gamma_{j,l} = 2^{(-j-J)a} m \quad \text{with} \quad m \in \mathbb{Z}^n$$

▶ Proof ideas, left-hand side:

- let

$$f_j^a(\mathbf{x}) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(\mathbf{x} - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad \mathbf{x} \in \mathbb{R}^n$$

- we obtain that  $\|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{\frac{1}{p}}$

and we get

$$\|f_j^a|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{1/p}$$

▶ Proof ideas, left-hand side:

• let

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(x - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n$$

• we obtain that  $\|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{\frac{1}{p}}$

and we get

$$\begin{aligned} \|f_j^a|L_p(\Gamma)\| &\geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{1/p} \\ \Rightarrow \|f_j^a|L_p(\Gamma)\| &\geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \quad \text{if} \quad \|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim 1 \end{aligned}$$

▶ Proof ideas, left-hand side:

- let

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(x - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n$$

- we obtain that  $\|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{\frac{1}{p}}$

and we get

$$\|f_j^a|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{1/p}$$

$$\Rightarrow \|f_j^a|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \quad \text{if} \quad \|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim 1$$

- let  $T : B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$  be an arbitrary linear operator with  $\text{rank } T \leq M_j - 1$

▶ Proof ideas, left-hand side:

• let

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(x - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n$$

• we obtain that  $\|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{\frac{1}{p}}$

and we get

$$\begin{aligned} \|f_j^a|L_p(\Gamma)\| &\geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{1/p} \\ \Rightarrow \|f_j^a|L_p(\Gamma)\| &\geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \quad \text{if} \quad \|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim 1 \end{aligned}$$

• let  $T : B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$  be an arbitrary linear operator with  $\text{rank } T \leq M_j - 1 \rightarrow \exists f_j^a : Tf_j^a = 0, \|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim 1$

▶ Proof ideas, left-hand side:

- let

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(x - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n$$

- we obtain that  $\|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{\frac{1}{p}}$

and we get

$$\|f_j^a|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{1/p}$$

$$\Rightarrow \|f_j^a|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \quad \text{if} \quad \|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim 1$$

- let  $T : B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$  be an arbitrary linear operator with  $\text{rank } T \leq M_j - 1 \rightarrow \exists f_j^a : Tf_j^a = 0, \|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim 1$
- $\|tr_\Gamma - T\| \geq \|tr_\Gamma f_j^a - 0|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})-j\frac{d}{p}}$

▶ Proof ideas, left-hand side:

- let

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(x - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n$$

- we obtain that  $\|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{\frac{1}{p}}$

and we get

$$\|f_j^a|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{1/p}$$

$$\Rightarrow \|f_j^a|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})} 2^{-j\frac{d}{p}} \quad \text{if} \quad \|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim 1$$

- let  $T : B_{pp}^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$  be an arbitrary linear operator with  $\text{rank } T \leq M_j - 1 \rightarrow \exists f_j^a : Tf_j^a = 0, \|f_j^a|B_{pp}^{s,a}(\mathbb{R}^n)\| \sim 1$

- $\|tr_\Gamma - T\| \geq \|tr_\Gamma f_j^a - 0|L_p(\Gamma)\| \geq c 2^{-j(s-\frac{n}{p})-j\frac{d}{p}}$   
 $a_{M_j}(tr_\Gamma) = \inf\{\|tr_\Gamma - T\| : \text{rank } T \leq M_{j-1}\} \geq c 2^{-j(s-\frac{n}{p})-j\frac{d}{p}}$

## Remark:

- Comparison with Farkas' results for entropy numbers

$$e_k(\text{tr}_\Gamma : B_{p_1 q}^{s + \frac{n-d}{p_1}, a}(\mathbb{R}^n) \longrightarrow L_{p_2}(\Gamma)) \sim ck^{-\frac{s}{d}}$$

where  $0 < p_1, p_2, q \leq \infty$ , and now let  $p_1 = p_2 = q = p$  and  $\delta = s + \frac{n-d}{p}$   
 we get

$$e_k(\text{tr}_\Gamma : B_{pp}^{\delta, a}(\mathbb{R}^n) \longrightarrow L_p(\Gamma)) \sim ck^{\frac{1}{d}(\frac{n}{p} - \delta) - \frac{1}{p}} \sim a_k(\text{tr}_\Gamma : B_{pp}^{\delta, a}(\mathbb{R}^n) \longrightarrow L_p(\Gamma)).$$

## Remark:

- Comparison with Farkas' results for entropy numbers

$$e_k(\text{tr}_\Gamma : B_{p_1 q}^{s + \frac{n-d}{p_1}, a}(\mathbb{R}^n) \longrightarrow L_{p_2}(\Gamma)) \sim ck^{-\frac{s}{\delta}}$$

where  $0 < p_1, p_2, q \leq \infty$ , and now let  $p_1 = p_2 = q = p$  and  $\delta = s + \frac{n-d}{p}$  we get

$$e_k(\text{tr}_\Gamma : B_{pp}^{\delta, a}(\mathbb{R}^n) \longrightarrow L_p(\Gamma)) \sim ck^{\frac{1}{d}(\frac{n}{p} - \delta) - \frac{1}{p}} \sim a_k(\text{tr}_\Gamma : B_{pp}^{\delta, a}(\mathbb{R}^n) \longrightarrow L_p(\Gamma)).$$

- In view of the isotropic result, Triebel 2004, if we restrict the outcome to the classical example of a compact  $d$ -set with  $0 < d < n$ , then we have the same result like in the anisotropic setting.

# Semi-elliptic operators

## Property of approximation numbers

Let  $H$  be a Hilbert space and let  $T \in L(H)$  be a compact, non-negative and self adjoint operator. Then the approximation numbers  $a_k(T)$  of  $T$  coincide with its eigenvalues.

# Semi-elliptic operators

Let  $A$  be the operator defined by

$$Au(x) = (-1)^{s_1} \frac{\partial^{2s_1} u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n} \frac{\partial^{2s_n} u(x)}{\partial x_n^{2s_n}} + u(x),$$

# Semi-elliptic operators

Let  $A$  be the operator defined by

$$Au(\mathbf{x}) = (-1)^{s_1} \frac{\partial^{2s_1} u(\mathbf{x})}{\partial \mathbf{x}_1^{2s_1}} + \cdots + (-1)^{s_n} \frac{\partial^{2s_n} u(\mathbf{x})}{\partial \mathbf{x}_n^{2s_n}} + u(\mathbf{x}),$$

where  $\frac{1}{s} = \frac{1}{n} \left( \frac{1}{s_1} + \cdots + \frac{1}{s_n} \right)$ ,  $s_i \in \mathbb{N}$ ,  $i = 1, \dots, n$ ,

# Semi-elliptic operators

Let  $A$  be the operator defined by

$$Au(x) = (-1)^{s_1} \frac{\partial^{2s_1} u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n} \frac{\partial^{2s_n} u(x)}{\partial x_n^{2s_n}} + u(x),$$

where  $\frac{1}{s} = \frac{1}{n} \left( \frac{1}{s_1} + \cdots + \frac{1}{s_n} \right)$ ,  $s_i \in \mathbb{N}$ ,  $i = 1, \dots, n$ ,

And let

$$tr^\Gamma = id_\Gamma \circ tr_\Gamma : W_2^{s,a}(\mathbb{R}^n) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n),$$

# Semi-elliptic operators

Let  $A$  be the operator defined by

$$Au(x) = (-1)^{s_1} \frac{\partial^{2s_1} u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n} \frac{\partial^{2s_n} u(x)}{\partial x_n^{2s_n}} + u(x),$$

where  $\frac{1}{s} = \frac{1}{n} \left( \frac{1}{s_1} + \cdots + \frac{1}{s_n} \right)$ ,  $s_i \in \mathbb{N}$ ,  $i = 1, \dots, n$ ,

And let

$$tr^\Gamma = id_\Gamma \circ tr_\Gamma : W_2^{s,a}(\mathbb{R}^n) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n),$$

where  $id_\Gamma : L_2(\Gamma) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n)$

# Semi-elliptic operators

Let  $A$  be the operator defined by

$$Au(x) = (-1)^{s_1} \frac{\partial^{2s_1} u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n} \frac{\partial^{2s_n} u(x)}{\partial x_n^{2s_n}} + u(x),$$

where  $\frac{1}{s} = \frac{1}{n} \left( \frac{1}{s_1} + \cdots + \frac{1}{s_n} \right)$ ,  $s_i \in \mathbb{N}$ ,  $i = 1, \dots, n$ ,

And let

$$tr^\Gamma = id_\Gamma \circ tr_\Gamma : W_2^{s,a}(\mathbb{R}^n) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n),$$

where  $id_\Gamma : L_2(\Gamma) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n)$  and  $tr_\Gamma : W_2^{s,a}(\mathbb{R}^n) \hookrightarrow L_2(\Gamma)$ .

## Theorem

Let  $\Gamma \subset \mathbb{R}^n$  be an anisotropic  $d$ -set with respect to the anisotropy  $a$ . Let  $0 < d < n$ ,  $\frac{n}{2} \geq s > \frac{n-d}{2}$ . Then

$$T = A^{-1} \circ tr^\Gamma$$

is a compact, non-negative self-adjoint operator in  $W_2^{s,a}(\mathbb{R}^n)$  with null space

$$N(T) = \{f \in W_2^{s,a}(\mathbb{R}^n) : tr_\Gamma f = 0\}.$$

Let  $(\lambda_k)_{k \in \mathbb{N}}$  be the sequence of all positive eigenvalues of  $T$ , repeated according to multiplicity and ordered by their magnitude. Then

$$\lambda_k \sim k^{-\frac{1}{d}(2s-n+d)}, \quad k \in \mathbb{N}.$$

## Remark:

- Comparison with Farkas' results for the operator  $A^{-1} \circ tr^\Gamma$

## Remark:

- Comparison with Farkas' results for the operator  $A^{-1} \circ tr^\Gamma$

## Remark:

- Comparison with Farkas' results for the operator  $A^{-1} \circ tr^\Gamma$

$$tr^\Gamma : B_{p1}^{\frac{2-d}{p}, a}(\mathbb{R}^2) \rightarrow B_{p\infty}^{-\frac{2-d}{p'}, a}(\mathbb{R}^2)$$

## Remark:

- Comparison with Farkas' results for the operator  $A^{-1} \circ tr^\Gamma$  if  $n = 2$ :

$$\lambda_k(A^{-1} \circ tr^\Gamma) \sim k^{-\frac{1}{d}(d+2t-2)}$$

## Remark:

- Comparison with Farkas' results for the operator  $A^{-1} \circ tr^\Gamma$  if  $n = 2$ :

$$\lambda_k(A^{-1} \circ tr^\Gamma) \sim k^{-\frac{1}{d}(d+2t-2)}$$

- In view of the isotropic result, Triebel 2004, for the operator

$$B_s = (id - \Delta)^{-s} \circ id^\mu$$

we have the same results like in the anisotropic case if we restrict the outcome to the classical example of a compact  $d$ -set with  $0 < d < n$  and  $n - d < 2s \leq n$ .

Thank you for your attention!