

Manifolds with bounded geometry

(X, g) - connected d -dimensional Riemannian manifold with Riemannian metric tensor g

X is called a manifold with bounded geometry if:

- (i) X is complete and connected.
- (ii) $r_{inj} = \inf\{x \in X : r_{inj}(x)\} > 0$,
- (iii) $|\nabla^k R| \leq C_k$, $k = 0, 1, 2, \dots$, (i.e. every covariant derivative of the Riemannian curvature tensor is bounded.)

Examples: compact manifolds, symmetric spaces, Lie groups with left (right) Riemannian structure.

Define the space of distribution $\mathcal{D}'(X) := \left(C_0^\infty(X)\right)'$ and identify functions with distributions via Riemannian measure vol.

The Laplace-Beltrami operator on X , $\Delta f = \operatorname{div} \nabla f$, is a negatively defined essentially self-adjoint operator.

$\mathcal{H}_t = e^{t\Delta}$, $t \geq 0$, – the heat semi-group;

$$\mathcal{H}_t f(x) = \int_X f(y) h_t(x, y) d \operatorname{vol}(y)$$

$$f = \sum_{\ell=0}^{k-1} \frac{1}{\ell!} (-\Delta)^\ell \mathcal{H}_1 f + \frac{1}{(k-1)!} \int_0^1 t^k (-\Delta)^k \mathcal{H}_t f \frac{dt}{t}$$

$$\mathcal{H}_{0,k} := \sum_{\ell=0}^{k-1} \frac{1}{\ell!} (-\Delta)^\ell \mathcal{H}_1$$

Function spaces on manifolds

(i) Sobolev spaces $1 < p < \infty$, $s \in \mathbb{R}$.

$$H_p^s(X) = \begin{cases} \{f : f = (I - \Delta)^{-s/2}h, h \in L_p(X)\}, & \text{if } s > 0, \\ L_p(X), & \text{if } s = 0, \\ \{f : f = (I - \Delta)^m h, h \in H_p^{2m+s}\}, \\ \quad m \in \mathbb{N}, 2m + s > 0, & \text{if } s < 0, \end{cases}$$

$$\|f\|_{H_p^s} = \begin{cases} \|h\|_p, & \text{if } s > 0 \\ \|h\|_{H_p^{2m+s}}, & \text{if } s < 0 \end{cases}$$

$$\|f\|_{H_p^s} \sim \|\mathcal{H}_{0,k}f\|_p + \left\| \left(\int_0^1 t^{(k-\frac{s}{2})2} \left| \frac{d^k}{dt^k} \mathcal{H}_t f(\cdot) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p$$

(ii) Besov spaces, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$.

$$B_{p,q}^s(X) = \{f \in \mathcal{S}'_\kappa(X) : \|f\|_B^{(k)} < \infty\},$$

$$\|f\|_B^{(k)} = \|\mathcal{H}_{0,k}f\|_p + \left(\int_0^1 t^{(k-\frac{s}{2})q} \left\| \frac{d^k}{dt^k} \mathcal{H}_t f \right\|_p^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

If $s > 0$ then in both cases $\|\mathcal{H}_{0,k}f\|_p$ can be replaced by $\|f\|_p$.

(iii) Characterization via uniform localization.

$\{B(x_j, r)\}_j$ - a uniformly locally finite covering of X by balls of radius $r < r_{inj}$,

$\{\varphi_j\}_j \subset C_o^\infty(X)$ - a corresponding resolution of unity,

$0 < p < \infty, 0 < q \leq \infty, p = q = \infty, s \in \mathbb{R}$

$$B_{p,p}^s(X) = \{f \in \mathcal{D}'(X) : \|f\|_{B_{p,p}^s(X)} = \left(\sum_j \|\varphi_j f \circ \exp_{x_j}\|_{B_{p,p}^s(\mathbb{R}^d)}^p \right)^{1/p} < \infty\}.$$

In the same way one can define $H_p^s(X)$ spaces.

(iv) Real interpolation

$$\begin{aligned} B_{p,q}^s(X) &= \left(B_{p,p}^{s_0}(X), B_{p,p}^{s_1}(X) \right)_{\theta,q} \\ &= \left(H_p^{s_0}(X), H_p^{s_1}(X) \right)_{\theta,q}, \end{aligned}$$

$s = \theta s_0 + (1 - \theta) s_1, s_1 \neq s_0 \in \mathbb{R}.$

Weighted function spaces on manifolds

Definition 1 (a) We call a function $w : X \rightarrow (0, \infty)$ an admissible weight on noncompact manifold X if:

$$w \in C^\infty(X),$$

$$\sup_{x \in X} \sup_{y \in B(x,1)} \frac{w(y)}{w(x)} < \infty,$$

$$\forall \alpha \in \mathbb{N}_0^d \quad \sup_{x \in X} \frac{|\nabla^\alpha w(x)|}{w(x)} < \infty.$$

(b) An admissible weight w is called radial with respect to $x_o \in X$ if there is a positive function $\tilde{w} : [0, \infty) \rightarrow [0, \infty)$ such that $w(x) \sim \tilde{w}(d(x_o, x))$

(c) $B_{p,q}^s(X, w) = \left\{ f \in \mathcal{D}'(X) : wf \in B_{p,q}^s(X) \right\}$
 and $\|f\|_{B_{p,q}^s(X, w)} = \|wf\|_{B_{p,q}^s(X)}$

We put

$$v_m = \text{vol} \left(B(x_o, m + \frac{1}{2}) \setminus B(x_o, m - \frac{1}{2}) \right), \quad m = 1, 2, \dots$$

Theorem 1 Let $0 < p_1, p_2, q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$. Let $w(x) = \tilde{w}(|x|)$ be a radial weight defined on manifold X .

(i) The embedding $B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_2, q_2}^{s_2}(X)$ holds if and only if

$$\sum_{m=1}^{\infty} \tilde{w}(m)^{-p^*} v_m < \infty \quad \text{if } p^* < \infty, \quad (1)$$

$$\text{or } \sup_m \tilde{w}(m)^{-1} < \infty \quad \text{if } p^* = \infty \quad (2)$$

and

$$s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \geq \frac{d}{p^*} \quad \text{if } q^* = \infty, \quad (3)$$

$$s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{d}{p^*} \quad \text{if } q^* < \infty, \quad (4)$$

where

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ \quad \text{and} \quad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+.$$

(ii) The embedding $B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_2, q_2}^{s_2}(X)$ is compact if and only if the conditions (1)–(4) hold and in addition

$$s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{d}{p^*} \quad \text{if } q^* = \infty, \quad (5)$$

and

$$\lim_{m \rightarrow \infty} \tilde{w}(m) = \infty \quad \text{if } p^* = \infty. \quad (6)$$

Wavelet frames on manifolds

$\{B(x_m, 1)\}_m$ - a uniformly locally finite covering of X .
 $\{\varphi_m\}$ - a resolution of unity corresponding to the covering. Then

$$\|f\|_{L_2(X)} \sim \left(\sum_m \|(\varphi_m f) \circ \exp_{x_m}^{-1}\|_{L_2(\mathbb{R}^d)}^2 \right)^{1/2}. \quad (7)$$

$(\varphi_m f) \circ \exp_{x_m} \in L_2(\mathbb{R}^d)$, $\text{supp}(\varphi_m f) \circ \exp_{x_m} \subset B_e(0, 1)$.

$\tilde{\phi}$ - an orthogonal scaling function on \mathbb{R} ,

$$\tilde{\phi} \in C^{N_1} \quad \text{and} \quad \text{supp } \tilde{\phi} \subset [-N_2, N_2] \quad , N_1, N_2 \in \mathbb{N}$$

$\tilde{\psi}$ - one corresponding wavelet.

Tensor product ansatz yields:

a scaling function ϕ and associated wavelets $\psi_1, \dots, \psi_{2^d-1}$ defined on \mathbb{R}^d . Then

$$\phi, \psi_i \in C^{N_1} \quad \text{and} \quad \text{supp } \phi, \text{supp } \psi_i \subset [-N_3, N_3]^d, .$$

We shall use the standard abbreviations

$$\phi_{j,\ell}(x) = 2^{jd/2} \phi(2^j x - \ell) \quad \text{and} \quad \psi_{i,j,\ell}(x) = 2^{jd/2} \psi_i(2^j x - \ell).$$

The family $\{\phi_{0,\ell}\} \cup \{\psi_{i,j,\ell}\}$ is an orthogonal basis in $L_2(\mathbb{R}^d)$.

$L_{0,0} = \{\ell \in \mathbb{Z}^d : B_e(0, 1) \cap \text{supp } \phi_{0,\ell} \neq \emptyset\}$ and

$L_{i,j} = \{\ell \in \mathbb{Z}^d : B_e(0, 1) \cap \text{supp } \psi_{i,j,\ell} \neq \emptyset\}$

$|L_{(0,j)}| \sim |L_{(i,j)}| \sim 2^{jd}$. Then

$$\begin{aligned} (\varphi_m f) \circ \exp_{x_m}^{-1} &= \sum_{\ell \in L_{0,0}} \langle (\varphi_m f) \circ \exp_{x_m}^{-1}, \phi_{0,\ell} \rangle_e \phi_{0,\ell} \\ &+ \sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} \sum_{\ell \in L_{i,j}} \langle (\varphi_m f) \circ \exp_{x_m}^{-1}, \psi_{i,j,\ell} \rangle_e \psi_{i,j,\ell}, \end{aligned}$$

where \langle, \rangle_e - the scalar product in $L_2(\mathbb{R}^d)$. Moreover

$$\begin{aligned} &\| (\varphi_m f) \circ \exp_{x_m}^{-1} \|_{L_2(\mathbb{R}^d)} \sim \\ &\left(\sum_{\ell \in L_{0,0}} | \langle (\varphi_m f) \circ \exp_{x_m}^{-1}, \phi_{0,\ell} \rangle_e |^2 \right)^{1/2} + \\ &+ \sum_{i=1}^{2^d-1} \left\{ \sum_{j=0}^{\infty} \sum_{\ell \in L_{i,j}} | \langle (\varphi_m f) \circ \exp_{x_m}^{-1}, \psi_{i,j,\ell} \rangle_e |^2 \right\}^{1/2}. \end{aligned}$$

We put

$$\Phi_{0,m,\ell}(x) = \varphi_m(x) (\sqrt{|g|}^{-1} \phi_{0,\ell})(\exp_{x_m}(x)), \quad \ell \in L_{0,0},$$

$$\Psi_{j,m,\ell}(x) = \varphi_m(x) (\sqrt{|g|}^{-1} \psi_{j,\ell})(\exp_{x_m}(x)), \quad \ell \in L_{i,j},$$

and extend the functions outside the ball $B(x_k, 1)$ by zero.

Then

$$\begin{aligned} \langle f, \Phi_{0,m,l} \rangle &= \langle (\varphi_m f) \circ \exp_{x_m}^{-1}, \phi_{0,l} \rangle_e, \\ \langle f, \Psi_{i,m,j,l} \rangle &= \langle (\varphi_m f) \circ \exp_{x_m}^{-1}, \psi_{i,j,l} \rangle_e. \end{aligned}$$

where \langle, \rangle is the scalar product of the space $L_2(X)$.

$$\begin{aligned} \|f\|_{L_2(X)} &\sim \\ &\left(\sum_m \sum_{\ell \in L_{0,0}} |\langle f, \Phi_{0,m,\ell} \rangle|^2 \right)^{1/2} + \\ &+ \sum_{i=1}^{2^d-1} \left\{ \sum_{j=0}^{\infty} \sum_m \sum_{\ell \in L_{i,j}} |\langle f, \Psi_{i,m,j,\ell} \rangle|^2 \right\}^{1/2}. \end{aligned}$$

Thus the family $\{\Phi_{0,m,\ell}\} \cup \{\Psi_{i,j,m,\ell}\}$, is a frame on the manifold X .

$\mathcal{X} = \{x_m\}$ be a discretization of the manifold X such that the balls $\{B(x_m, 1)\}_{m \in \mathcal{X}}$ form a covering of X and moreover $B(x_{m_1}, \frac{1}{2})$ and $B(x_{m_2}, \frac{1}{2})$ are disjoint if $x_{m_1} \neq x_{m_2}$.

Theorem 2 *Let w be an admissible weight on manifold X with bounded geometry. Let $\{\Phi_{0,m,\ell}\} \cup \{\Psi_{i,j,m,\ell}\}$ be the frame on X defined above. Let $1 \leq p, q \leq \infty$ and let $0 < s < N_1$. Then a function $f \in L_p(w)$ belongs to $B_{p,q}^s(w)$ if and only if*

$$\begin{aligned} \|f\|_{B_{p,q}^s(X,w)}^{\clubsuit} = & \left(\sum_m \sum_{\ell \in L_{0,0}} |\langle f, \Phi_{0,m,\ell} \rangle w(x_m)|^p \right)^{1/p} + \\ & + \sum_{i=1}^{2^d-1} \left(\sum_{j=0}^{\infty} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))} \right)_q \\ & \left(\sum_m \sum_{\ell \in L_{i,j}} |\langle f, \Psi_{i,j,m,\ell} \rangle w(x_m)|^p \right)^{q/p} \right)^{1/q} < \infty. \end{aligned}$$

Furthermore, $\|f\|_{B_{p,q}^s(X,w)}^{\clubsuit}$ may be used as an equivalent norm on $B_{p,q}^s(X,w)$.

In consequence we can deal with the following sequence spaces

$$\mathfrak{b}_{p,q}^s(\mathcal{X}, w) := \left\{ \lambda = \{\lambda_{i,j,m,\ell}\}_{i,j,m,\ell} : \right. \\ \left. \lambda_{i,j,m,\ell} \in \mathbb{C}, \|\lambda\|_{\mathfrak{b}_{p,q}^s(\mathcal{X}, w)} < \infty \right\}$$

where

$$\|\lambda\|_{\mathfrak{b}_{p,q}^s(\mathcal{X}, w)} := \\ \left(\sum_m \sum_{\ell \in L_{0,0}} |\lambda_{0,0,m,\ell} w(x_m)|^p \right)^{1/p} + \\ + \sum_{i=1}^{2^d-1} \left(\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_m \sum_{\ell \in L_{i,j}} |\lambda_{i,j,m,\ell} w(x_m)|^p \right)^{q/p} \right)^{1/q},$$

and

$$b_{p,q}^s(\mathcal{X}, w) := \left\{ \lambda = \{\lambda_{j,m,\ell}\}_{j,m,\ell} : \lambda_{j,m,\ell} \in \mathbb{C}, \right. \\ \left. \|\lambda\|_{b_{p,q}^s(\mathcal{X}, w)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_m \sum_{\ell=1}^{2^{dj}} |\lambda_{j,m,\ell} w(x_m)|^p \right)^{q/p} \right)^{1/q} < \infty \right\}.$$

We put $v(t) = \text{vol } B(x_o, t)$ and $|x| = d(x_o, x)$, $x_o, x \in X$.

Entropy numbers of embeddings.

Theorem 3 *Let $0 < p_1, p_2, q_1, q_2 \leq \infty$ and $s_1, s_2 \in \mathbb{R}$ satisfy assumptions (1)–(6). Let $w(x) = \tilde{w}(|x|)$ be a radial weight defined on manifold X . We put $\delta = s_1 - \frac{d}{p_1} - s_2 + \frac{d}{p_2}$.*

(1) *Let $\tilde{w}(v^{-1}(t)) \sim (1+t)^\alpha$, $\alpha > 0$. Then*

$$e_k \left(B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_2, q_2}^{s_2}(X) \right) \sim k^{-\left(\min(\alpha, \frac{\delta}{d}) + \frac{1}{p_1} - \frac{1}{p_2}\right)}$$

if $\alpha \neq \frac{\delta}{d}$,

$$e_k \left(B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_2, q_2}^{s_2}(X) \right) \sim k^{-\frac{s_1 - s_2}{d}} (1 + \log k)^\tau$$

if $\alpha = \frac{\delta}{d}$ and $\tau = \frac{s_1 - s_2}{d} + \frac{1}{q_1} - \frac{1}{q_2} > 0$,

$$e_k \left(B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_2, q_2}^{s_2}(X) \right) \sim k^{-\frac{s_1 - s_2}{d}}$$

if $\alpha = \frac{\delta}{d}$ and $\tau = \frac{s_1 - s_2}{d} + \frac{1}{q_1} - \frac{1}{q_2} < 0$.

(2) *If for any $\alpha > 0$ there is $C > 0$ such that $\tilde{w}(v^{-1}(t)) \geq C(1+t)^\alpha$, then*

$$e_k \left(B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_2, q_2}^{s_2}(X) \right) \sim k^{-\frac{s_1 - s_2}{d}}.$$

Theorem 4 We assume that $\tilde{w}(v^{-1}(t)) \sim 1$ for $t \leq 1$ and $\tilde{w}(v^{-1}(t)) \sim \varphi(t)$ for $t \geq 1$, where φ is an non-decreasing function such that there exist $c > 0$ and $\lambda > 0$ such that

$$\frac{\varphi(t)}{\varphi(\tau)} \leq c \left(\frac{1 + \log t}{1 + \log \tau} \right)^\alpha \quad \text{if} \quad 1 \leq \tau \leq t < \infty. \quad (8)$$

(1) If the function φ satisfies the assumption

$$\frac{\varphi(t)}{\varphi(\tau)} \leq c \left(\frac{1 + \log t}{1 + \log \tau} \right)^{1/p} \quad \text{if} \quad 1 \leq s \leq t < \infty,$$

$p_1 < p_2$ and $\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}$, then

$$e_k \left(B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_2, q_2}^{s_2}(X) \right) \sim \frac{1}{\varphi(2^k)}$$

On the other hand if the function φ satisfies the assumption

$$\frac{\varphi(t)}{\varphi(\tau)} \geq c \left(\frac{1 + \log t}{1 + \log \tau} \right)^{1/p} \quad \text{if} \quad 1 \leq \tau \leq t < \infty$$

then

$$e_k \left(B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_2, q_2}^{s_2}(X) \right) \sim \frac{\log^{1/p} k}{k^{1/p} \varphi(k)}.$$

Theorem 5 *Let $p_1 = p_2$. We assume that $\tilde{w}(v^{-1}(t)) \sim 1$ for $t \leq 1$ and $\tilde{w}(v^{-1}(t)) \sim \varphi(t)$ for $t \geq 1$, where φ is an non-decreasing function such that $\varphi(2t) \sim \varphi(t)$ and*

$$\frac{\varphi(t)}{\varphi(\tau)} \leq C \left(\frac{t}{\tau} \right)^\alpha \quad 1 \leq \tau \leq t < \infty \quad \alpha < \delta = s_1 - s_2$$

then

$$e_k \left(B_{p_1, q_1}^{s_1}(X, w) \hookrightarrow B_{p_1, q_2}^{s_2}(X) \right) \sim \frac{1}{\varphi(k)}.$$

Negative spectrum

Hyperbolic case: X^d - d-dimensional symmetric space of noncompact type.

$b_p = \frac{4}{pp'}|\rho|^2$ a bottom of L_p -spectrum of $-\Delta$.

For $\theta \in (0, 1)$ we put $\Delta_\theta = -\Delta - \theta|\rho|^2$. If $0 \leq \theta < \frac{4}{pp'}$ then Δ_θ is an invertible in L_p , $1 < p < \infty$, and its inverse is bounded.

$$H_{s,\beta,\theta} = \Delta_\theta^{s/2} - \beta V, \quad s > 0, \beta > 0$$

$V \geq 0$ is a real potential. We will assume that $H_{s,\beta,\theta}$ is a self-adjoint operator in $L_2(X)$ with the domain $H_2^s(X)$.

We are interested in a number of “negative” eigenvalues of $H_{s,\beta,\theta}$.

$$N_\gamma(H_{s,\beta,\theta}) = \#\left\{ \lambda < \gamma : H_{s,\beta,\theta} f = \lambda f, 0 \neq f \in L_2(X) \right\},$$

$$\gamma < (1 - \theta)|\rho|^2.$$

Theorem 6 *Let w be an admissible radial weight on X such that the conditions (1) and (6) are satisfied. Let $H_{s,\beta,\theta}$ be an operator defined as above. We assume that the potential V is a real measurable function on the manifold X such that $V \in L_\infty(X, w)$.*

(1) *Then the operator $H_{s,\beta,\theta}$ is self-adjoint and*

$$\sigma_{ess}(H_{s,\beta,\theta}) = \sigma_{ess}(\Delta_\theta) = \langle (1 - \theta)|\rho|^2, \infty \rangle.$$

(2) *If in addition w is such that*

$$w(v^{-1}(t)) \sim \varphi(t), \quad t > 1,$$

where φ is an increasing function with $\varphi(2t) \sim \varphi(t)$, then

$$(a) \quad N_\gamma(H_{s,\beta,\theta}) \leq C_\gamma \varphi^{-1} \left(\beta \|V\|_{L_\infty(X, w)} \right),$$

provided that

$$\frac{\varphi(t)}{\varphi(\tau)} \leq C \left(\frac{t}{\tau} \right)^\alpha, \quad 1 \leq \tau \leq t < \infty, \quad \alpha < s,$$

$$(b) \quad N_\gamma(H_{s,\beta,\theta}) \leq C_\gamma \left(\beta \|V\|_{L_\infty(X, w)} \right)^{d/s},$$

provided that

$$\varphi(t) \geq Ct^\alpha, \quad t > 1, \quad \alpha > s.$$