Some Remarks on Multi-Parameter Processes

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Barndorff-Nielsen, Pedersen, Sato:
"Multivariate Subordination, Self-decomposability and Stability", 2001

Khoshnevisan: "Multiparameter Processes", 2002

Pedersen, Sato:
"Relations between cone-parameter Levy processes and convolution semigroups", 2004

Some Motivation

- From a pure analytic point of view the question arises: How can strongly continuous semigroups T_t be extended to the case of an N-dimensional parameter t?
- analytic $\longleftrightarrow$ probabilistic: How do N-parameter semigroups correspond to processes with N-dimensional time-parameter.

Definition 1. *Let* $\mathsf{q} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ *be continuous and negative definite with respect to the second variable, i.e.* $\mathsf{q}(x, .) : \mathbb{R}^n \longrightarrow \mathbb{C}$ *is continuous and negative definite. On* $\mathsf{C}_0^\infty(\mathbb{R}^n)$ *we define the* **pseudo-differential operator** $\mathsf{q}(x, \mathsf{D})$ *by*

$$\mathsf{q}(x, \mathsf{D})\mathsf{u}(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathsf{e}^{\mathsf{i}x\xi} \mathsf{q}(x, \xi)\hat{\mathsf{u}}\xi \, \mathsf{d}\xi.$$

We call q *the* **symbol** *of the pseudo-differential operator* $\mathsf{q}(x, \mathsf{D})$.

N-Parameter Markov-Semigroups of Operators

Definition 2. *An* **N-parameter convolution semigroup of sub-probability measures** $(\mu_t)_{t \succeq 0}, t \epsilon \mathbb{R}_+^N$, *on* $\mathbb{R}^n$ *is defined by*

$$\begin{aligned}
&(i) && \mu_t(\mathbb{R}^n) \leq 1 && \textit{for all } t \succeq 0, \\
&(ii) && \mu_s * \mu_t = \mu_{s+t} && \textit{for all } s, t \succeq 0, \\
& && \textit{and } \mu_0 = \varepsilon_0, && \\
&(iii) && \mu_t \longrightarrow \varepsilon_0 \textit{ vaguely as } t \longrightarrow 0. &&
\end{aligned}$$

Example 3. *Let* $(\mu_{t_1})_{t_1 \geq 0}$ *and* $(\nu_{t_2})_{t_2 \geq 0}$ *be 1-parameter convolution semigroups of bounded Borel measures on* $\mathbb{R}^n$*, then*

$$\eta_{t_1,t_2} := \mu_{t_1} \otimes \nu_{t_2} \qquad \text{for all } t_1, t_2 \geq 0$$

defines a two-parameter convolution semigroup of Borel measures on $\mathbb{R}^{2n}$.

Definition 4. *A. An N-parameter family* $(\mathsf{T}_t)_{t\succeq 0}$, $t\epsilon\mathbb{R}_+^N$, *of bounded linear operators* $\mathsf{T}_t : X \longrightarrow X$ *with*

- $\mathsf{T}_0 = id$
- $\mathsf{T}_{s+t} = \mathsf{T}_s \circ \mathsf{T}_t$ *for all* $s, t\epsilon\mathbb{R}_+^N$.

is called an **N-parameter semigroup of operators***.*

B. We call it **strongly continuous** *if for all* $x\epsilon X$

$$\lim_{t\longrightarrow 0} \|\mathsf{T}_t \mathsf{u} - \mathsf{u}\|_X = 0.$$

C. The semigroup $(\mathsf{T}_t)_{t\succeq 0}$ *is a* **contraction** *semigroup, if for all* $t \succeq 0$

$$\|\mathsf{T}_t\| \leq 1.$$

Example 5. *Let* $(\mu_{t_1})_{t_1 \geq 0}$ *and* $(\nu_{t_2})_{t_2 \geq 0}$ *be two-parameter convolution semigroups on* $\mathbb{R}^n$. *On the Banach space* $(\mathsf{C}_\infty(\mathbb{R}^{2n}), \|.\|_\infty)$ *we define for all* $t \in \mathbb{R}^2_+$ *the operator*

$$\mathsf{T}_t \mathsf{u}(x) := \int_{\mathbb{R}^{2n}} \mathsf{u}(x-y)(\mu_{t_1} \otimes \nu_{t_2})(\mathrm{d}y). \tag{1}$$

Then $(\mathsf{T}_t)_{t \succeq 0}$ *is a strongly continuous contraction semigroup, which is positivity preserving.*

Definition 6. *A strongly continuous* N*-parameter contraction semigroup of operators on* $(\mathsf{C}_\infty(\mathbb{R}^{2n}), \|.\|_\infty)$ *which is positivity preserving is called in* N*-parameter* **Feller semigroup***.*

Definition 7. *Let* $(\mathsf{T}_t)_{t\succeq 0}$ *be an N-parameter semigroup of operators, for* $j = 1, \ldots, N$ *we define its* **j-th marginal** *(1-parameter)* **semigroup** *by*

$$\mathsf{T}^{(j)}_{t_j} := \mathsf{T}_{t_j \cdot \mathsf{e}_j} \qquad \text{for all } t_j \geq 0.$$

Remark 8. *With this definition every* $N-$*parameter semigroup can be represented as a composition of its marginal semigroups:*

$$\mathsf{T}_t = \mathsf{T}^{(1)}_{t_1} \circ \mathsf{T}^{(2)}_{t_2} \circ \ldots \circ \mathsf{T}^{(N)}_{t_N}.$$

The semigroups are commutating, i.e.

$$\left[\mathsf{T}^{(i)}_{t_i}, \mathsf{T}^{(j)}_{t_j}\right] = 0, \quad \text{for all } i, j = 1, \ldots, N$$

and for each marginal semigroup $\mathsf{T}^{(j)}_{t_j}$ *we denote its generator by* $\mathsf{A}^{(j)}$.

Example 9. *Consider two independent n-dimensional Brownian motions $B^{(1)}$ and $B^{(2)}$ and define for $t \in \mathbb{R}_+^2$*

$$\mathsf{X}_t = \mathsf{B}^{(1)}_{t_1} + \mathsf{B}^{(2)}_{t_2},$$

then X_t is a 2-parameter process, called the n-dimensional additive Brownian motion and with

$$\mathsf{T}_t\mathsf{u}(x) := (2\pi\{t_1 + t_2\})^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathsf{e}^{-\frac{\|y-x\|^2}{2\{t_1+t_2\}}} \mathsf{u}(y)\,\mathsf{d}y,$$

we have

$$\mathbb{E}^x\left[\mathsf{u}(\mathsf{X}_t)\right] = \mathsf{T}_t\mathsf{u}(x)$$

and

$$\mathbb{P}^x\left(\mathsf{X}_t \in \mathsf{A}\right) = \mathsf{T}_t\chi_{\mathsf{A}}(x).$$

Lemma 10. *For* $u\epsilon D(A^{(1)} \circ ... \circ A^{(N)})$ *and* $(T_t)_{t\succeq 0}$ *as above we have*

$$\frac{\partial^N}{\partial_{t_1}...\partial_{t_N}} T_t u = A^{(1)} \circ ... \circ A^{(N)} \circ T_t u.$$

Theorem 11. *Let* $(T_t)_{t\succeq 0}, t\epsilon\mathbb{R}_+^N$, *be a strongly continuous N-parameter contraction semigroup on a Banach space* $(X, \|.\|_X)$ *and* $(\eta_s)_{s\succeq 0}, s\epsilon\mathbb{R}_+^M$, *be an M-parameter convolution semigroup on* $\mathbb{R}^N$ *with* $\operatorname{supp}\eta_s \subset [0,\infty)^N$, *for all* $s\epsilon\mathbb{R}_+^M$. *Now, we define* $(T_s^\eta)_{s\succeq 0}$ *using the M-dimensional Bochner integral:*

$$T_s^\eta u := \int_{\mathbb{R}_+^M} T_t u \, \eta_s(dt). \tag{2}$$

Then the integral is well-defined and $(T_s^\eta)_{s\succeq 0}$ *is a strongly continuous contraction semigroup on X and is called the subordinate (in the sense of Bochner) to* $(T_t)_{t\succeq 0}$ *with respect to the convolution semigroup* $(\eta_s)_{s\succeq 0}$.

Corollary 12. *Let* $(T_t)_{t\succeq 0}$ *be either an N-parameter Feller semigroup on* $\mathsf{C}_\infty(\mathbb{R}^n;\mathbb{R})$ *or an N-parameter sub-Markovian semigroup on* $L^p(\mathbb{R}^n;\mathbb{R}), 1 \leq p \leq \infty$, *further let* $(\eta_s)_{s\succeq 0}$ *be an N-parameter convolution semigroup on* $\mathbb{R}_+^M$ *with* $\operatorname{supp}\eta_s \subset \mathbb{R}_+^N$, *i.e. with positive support. Then* $(T_s^\eta)_{s\succeq 0}$ *is an M-parameter Feller or sub-Markovian semigroup, respectively.*

Remark 13. *Especially, subordinating with respect to* $(\varepsilon_{\tau e_j})_{\tau\in\mathbb{R}^+}$ *we get the j-th marginal semigroup, i.e.*

$$\mathsf{T}_\tau^{(j)}u := \int_{\mathbb{R}_+^N} \mathsf{T}_t u\, \varepsilon_{\tau e_j}(dt).$$

Example 14. *Considering two pseudo-differential operators* $\mathsf{q}_1(x, \mathsf{D})$ *and* $\mathsf{q}_2(x, \mathsf{D})$ *on* $\mathcal{S}(\mathbb{R}^2)$, *then*

$$[\mathsf{q}_1(x, \mathsf{D}), \mathsf{q}_2(x, \mathsf{D})] = 0$$

holds for example, if

$$\mathsf{q}_1(x, \mathsf{D}) = \tilde{\mathsf{q}}_1(x_1, \mathsf{D}_{x_1})$$

and

$$\mathsf{q}_2(x, \mathsf{D}) = \tilde{\mathsf{q}}_2(x_2, \mathsf{D}_{x_2}),$$

with $\tilde{\mathsf{q}}_1(x_1, \mathsf{D}_{x_1})$ *and* $\tilde{\mathsf{q}}_2(x_2, \mathsf{D}_{x_2})$ *being pseudo-differential operators on* $\mathcal{S}(\mathbb{R})$.

Beyond N-Parameter Markov-Semigroups
- A Case Study

Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a continuous negative definite function. We consider

$$\phi_{s,t}(\xi) := (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-st\psi(\xi)},$$

which is for $s, t \geq 0$ a continuous and positive definite function, i.e. by Bochner's Theorem each $\phi_{s,t}$ is the Fourier transform of a measure on R^n. This leads us to the definition of the following **two-parameter family of measures** $(\mu_{s,t})_{s,t \geq 0}$:

$$\widehat{\mu}_{s,t}(\xi) := (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-st\psi(\xi)}, \tag{3}$$

and to the definition of a **two-parameter family of operators**

$$\mathsf{T}_{s,t}\mathsf{u}(x) := \int_{\mathbb{R}^n} \mathsf{u}(x-y)\,\mu_{s,t}(\mathrm{d}y).$$

Fix $s = s_0 \neq 0$, then

- $(\mu_{s_0,t})_{t\geq 0}$ is a convolution semigroup in t

- $\left(\mathsf{T}_{s_0,t}\right)_{t\geq 0}$ a semigroup of operators w.r.t. t

and for $s_0 = 0$ the operator $\mathsf{T}_{s_0,t}$ is the identity.

The Bernstein function $\mathsf{f}_1(x) = x^{1/2}$ is associated with the convolution semigroup

$$\eta_r^{(1)} = \mathsf{h}_r(.)\lambda^{(1)},\ \mathsf{h}_r(x) = \chi_{(x>0)}\frac{1}{\sqrt{2\pi}} \cdot r \cdot x^{-\frac{3}{2}} \cdot \mathsf{e}^{-\frac{r^2}{4x}}.$$

The Bernstein function $\mathsf{f}_2(x) = \log(1+x)$ is associated with the convolution semigroup

$$\eta_r^{(2)} = \mathsf{g}_r(.)\lambda^{(1)},\ \mathsf{g}_r(x) = \chi_{(x>0)}\frac{1}{\Gamma(r)}x^{r-1}\mathsf{e}^{-x}.$$

Subordinating $(\mu_{s,t})_{s,t\geq 0}$ by the Bernstein function $\mathsf{f}(x) = x^{1/2}$ with respect to parameter s gives us a new family of measures $(\nu_{r,t})_{r,t\geq 0}$, which can be characterized by its Fourier transform

$$\widehat{\nu}_{r,t}(\xi) = \mathsf{e}^{-rt^{1/2}[\psi(\xi)]^{1/2}}.$$

Then subordinating $(\nu_{r,t})_{r,t\geq 0}$ by $\mathsf{g}(x) = \log(1+x)$, which is a Bernstein function, with respect to parameter t leads to another family $(\tau_{r,p})_{r,t\geq 0}$ of measures:

$$\begin{aligned}
\widehat{\tau}_{r,p}(\xi) \;=\; & \frac{-\Gamma(2p)}{(\Gamma(p))^2} \cdot 2^{1-2p} \cdot [\psi(\xi)]^{(1/2)} \cdot \sqrt{\pi} \\
& \times r \cdot {}_1\mathsf{F}_1\left(\frac{1}{2}+p; \frac{3}{2}; \frac{1}{4} r^2 \psi(\xi)\right) \\
& + \frac{\Gamma(2p) \cdot 2^{1-2p}}{\Gamma(p)\Gamma(p+\frac{1}{2})} \cdot \sqrt{\pi} \\
& \times {}_1\mathsf{F}_1\left(p; \frac{1}{2}; \frac{1}{4} r^2 \psi(\xi)\right),
\end{aligned}$$

here ${}_1\mathsf{F}_1$ is the confluent hypergeometric function. The following table gives $\widehat{\tau}_{r,p}$, if $(\mu_{s,t})_{s,t\geq 0}$ is subordinated in the first step by $\mathsf{f}(x) = x^{1/n}$.

$\frac{1}{2}$	$\frac{\mathrm{e}^{r^2\psi/4}}{\Gamma(p)}\left(\Gamma(p)\ \ {}_1F_1\left(\frac{1}{2}-p;\frac{1}{2};-\frac{r^2\psi}{4}\right)\ -r\sqrt{\psi}\Gamma(\frac{1}{2}+p)\ \ {}_1F_1\left(1-p;\frac{3}{2};-\frac{r^2\psi}{4}\right)\right)$
$\frac{1}{3}$	$\frac{1}{2\Gamma(p)}\Big(2\Gamma(p)\ \ {}_1F_2(p;\frac{1}{3},\frac{2}{3};-\frac{r^3\psi}{27})\ -2r\psi^{1/3}\Gamma(\frac{1}{3}+p)\ \ {}_1F_2(\frac{1}{3}+p;\frac{2}{3},\frac{4}{3};-\frac{r^3\psi}{27})$ $+r^2\psi^{2/3}\Gamma(\frac{2}{3}+p)\ \ {}_1F_2(\frac{2}{3}+p;\frac{4}{3},\frac{5}{3};-\frac{r^3\psi}{27})\Big)$
$\frac{1}{4}$	${}_1F_3(p;\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{r^4\psi}{256})\ -\frac{\Gamma(\frac{1}{4}+p)}{\Gamma(p)}r\psi^{1/4}\ \ {}_1F_3(\frac{1}{4}+p;\frac{1}{2},\frac{3}{4},\frac{5}{4};\frac{r^4\psi}{256})$ $+\frac{\Gamma(\frac{1}{2}+p)}{2\Gamma(p)}r^2\psi^{1/2}\ \ {}_1F_3(\frac{1}{2}+p;\frac{3}{4},\frac{5}{4},\frac{3}{2};\frac{r^4\psi}{256})$ $+\frac{\Gamma(\frac{3}{4}+p)}{6\Gamma(p)}r^3\psi^{3/4}\ \ {}_1F_3(\frac{3}{4}+p;\frac{5}{4},\frac{3}{2},\frac{7}{4};\frac{r^4\psi}{256})$
$\frac{1}{5}$	${}_1F_4(p;\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};-\frac{r^5\psi}{3125})\ -\frac{\Gamma(\frac{1}{5}+p)}{\Gamma(p)}r\psi^{1/5}\ \ {}_1F_4(\frac{1}{5}+p;\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{6}{5};-\frac{r^5\psi}{3125})$ $+\frac{\Gamma(\frac{2}{5}+p)}{2\Gamma(p)}r^2\psi^{2/5}\ \ {}_1F_4(\frac{2}{5}+p;\frac{3}{5},\frac{4}{5},\frac{6}{5},\frac{7}{5};-\frac{r^5\psi}{3125})$ $+\frac{\Gamma(\frac{3}{5}+p)}{6\Gamma(p)}r^3\psi^{3/5}\ \ {}_1F_4(\frac{3}{5}+p;\frac{4}{5},\frac{6}{5},\frac{7}{5},\frac{8}{5};-\frac{r^5\psi}{3125})$ $+\frac{\Gamma(\frac{4}{5}+p)}{24\Gamma(p)}r^5\psi\ \ {}_1F_4(\frac{4}{5}+p;\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5};-\frac{r^5\psi}{3125})$

We start again with the two-parameter family defined earlier, now we subordinate by f$(x) = x^{1/2}$ with respect to s and t. Using the Meijer-G-function we can represent $\widehat{\tau}_{r,p}$ as follows

$$\widehat{\tau}_{r,p}(\xi) = \frac{r \cdot [\psi(\xi)]^{1/2}}{4\pi} \, \mathsf{G}_{0,3}^{3,0}\left(\begin{array}{cccc} - & - & - & \frac{1}{16}p^2 r^2 \psi \\ -\frac{1}{2} & 0 & 0 & \end{array} \right)$$

$\frac{1}{2}$	$\frac{pr}{4\pi} \times \sqrt{\psi} \;\; \mathrm{G}^{3,0}_{0,3}\left(\begin{matrix} \\ -\frac{1}{2} & 0 & 0 \end{matrix} \frac{1}{16}p^2r^2\psi \right)$
$\frac{1}{3}$	$\frac{1}{4\pi} \;\; {}_0F_3\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; \frac{1}{108}p^2r^3\psi\right) - 2\sqrt{3}r\sqrt[3]{p^2\psi}\Gamma\left(\frac{1}{3}\right)^2 \;\; {}_0F_3\left(\frac{2}{3}, \frac{5}{6}, \frac{4}{3}; \frac{1}{108}p^2r^3\psi\right)$ $+8\pi\sqrt{p^2r^3\psi} \;\; {}_0F_3\left(\frac{5}{6}, \frac{7}{6}, \frac{3}{2}; \frac{1}{108}p^2r^3\psi\right) + 3\sqrt{\pi}(2p^2r^3\psi)^(2/3)\Gamma\left(\frac{5}{6}\right) \;\; {}_0F_3\left(\frac{7}{6}, \frac{4}{3},\right.$
$\frac{1}{4}$	$\frac{pr^2\sqrt{\psi}}{32\sqrt{2}\pi^2} \times \;\; \mathrm{G}^{5,0}_{0,5}\left(\begin{matrix} \\ -\frac{1}{2} & -\frac{1}{4} & 0 & 0 & \frac{1}{4} \end{matrix} ; \frac{1}{1024}p^2r^4\psi \right)$
$\frac{1}{5}$	$\frac{1}{2\sqrt{\pi}}(5p(\frac{2}{5}\sqrt{\frac{\pi}{p^2}} \;\; {}_0F_5\left(\frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}; \frac{p^2r^5\psi}{12500}\right)$ $-p^{-3/5}\sqrt{\frac{10}{5\pi+\sqrt{5}\pi}}r\psi^{1/5}\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{6}{5}\right) \;\; {}_0F_5\left(\frac{2}{5}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{6}{5}; \frac{p^2r^5\psi}{12500}\right)$ $+\frac{1}{\sqrt{10(5-\sqrt{5})\pi}}\frac{r^2\psi^{2/5}}{p^{1/5}}\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right) \;\; {}_0F_5\left(\frac{3}{5}, \frac{4}{5}, \frac{9}{10}, \frac{6}{5}, \frac{7}{5}; \frac{p^2r^5\psi}{12500}\right)$ $-\frac{8\sqrt{\pi r^5\psi}}{15} \;\; {}_0F_5\left(\frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}, \frac{3}{2}; \frac{p^2r^5\psi}{12500}\right)$ $\frac{r^3\psi^{3/5}}{30}\left(\frac{p}{2}\right)^{1/5}\Gamma(-\frac{1}{10}) \;\; {}_0F_5\left(\frac{4}{5}, \frac{11}{10}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}; \frac{p^2r^5\psi}{12500}\right)$ $\frac{r^4\psi^{4/5}}{120}\left(\frac{p}{2}\right)^{3/5}\Gamma\left(-\frac{3}{10}\right) \;\; {}_0F_5\left(\frac{6}{5}, \frac{13}{10}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}; \frac{p^2r^5\psi}{12500}\right)))$

Constructing a Process

Starting with

$$\widehat{\mu}_t(\xi) = (2\pi)^{-n/2}\mathrm{e}^{-t_1 t_2 \psi(\xi)}, \text{ for all } t = (t_1, t_2) \in \mathbb{R}^2_+,$$

with ψ continuous, negative definite and being zero at the origin, we construct a projective family of probability measures over $\mathbb{R}^n$. Let

$$u_1 \preceq u_2 \preceq \ldots \preceq u_k$$

and

$$\{u_1, u_2, \ldots, u_k\} = \mathsf{K} \in \mathcal{H},$$

for the set $\mathcal{H}$ of all finite subset of $\mathbb{R}^2_+$.

For K we define the measure P_K on $(\mathbb{R}^n)^k$ by

$$\begin{aligned} &P_K(A_1 \times A_2 \times \ldots \times A_k) = \\ &= \int_{\mathbb{R}^n} \int_{A_1} \cdots \int_{A_k} p(u_{k-1}, u_k, x_{k-1}, dx_k) \cdots \\ &\times p(u_1, u_2, x_1, x_2) \cdot p(0, u_1, x, dx_1) \nu(dx), \end{aligned}$$

for all $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^n)$.
Here

$$p(u_j, u_{j+1}, x_j, A) = \mu_{u_j, u_{j+1}}(A - x_j),$$

for all $A \in \mathcal{B}^{(n)}$. Where $\mu_{u_j, u_{j+1}}(A - x_j)$ is defined by:

$$\widehat{\mu}_{u_j, u_{j+1}}(\xi) = (2\pi)^{-\frac{n}{2}} e^{-(u_{j+1}^{(1)} u_{j+1}^{(2)} - u_j^{(1)} u_j^{(2)}) \psi(\xi)}.$$

This family is projective, and applying Kolomogorov's Theorem it follows the existence of a probability measure P on $\mathcal{B}^{(n)}$ satisfying

$$\pi_{\mathsf{K}}(\mathsf{P}) = \mathsf{P}_{\mathsf{K}}, \text{ for all } \mathsf{K} \in \mathcal{H},$$

and the existence of a stochastic process with state space $\mathbb{R}^n$, whose finite-dimensional distributions are given by $(\mathsf{P}_{\mathsf{K}})_{\mathsf{K}\in\mathcal{H}}$.

More general, we can consider $(2\pi)^{-\frac{n}{2}}\mathsf{e}^{-\mathsf{k}(t,\xi)}$, where we have to assumme, that for fixed $t \in \mathbb{R}_+^N$ $k(t,.)$ is a continuous negative definite function and for all $t \succeq s$ and $\xi \in \mathbb{R}^n$

$$\mathsf{k}(t,\xi) \geq \mathsf{k}(s,\xi).$$