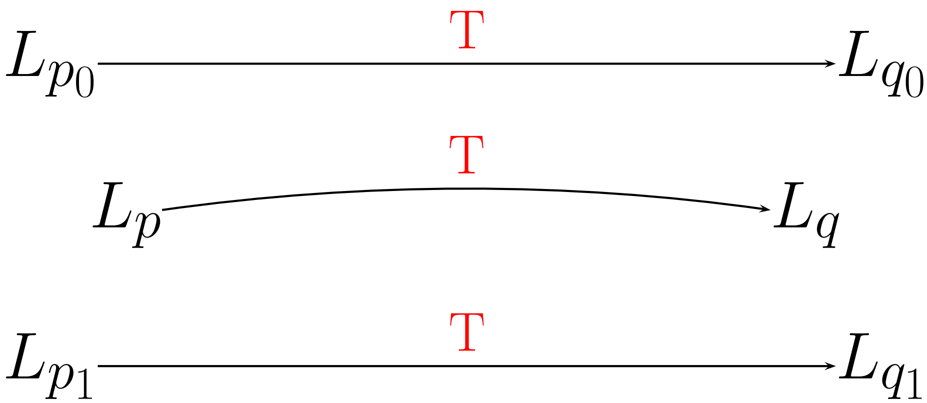
Interpolation of entropy numbers

and of

the measure of non-compactness

Riesz-Thorin Theorem.



$1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$

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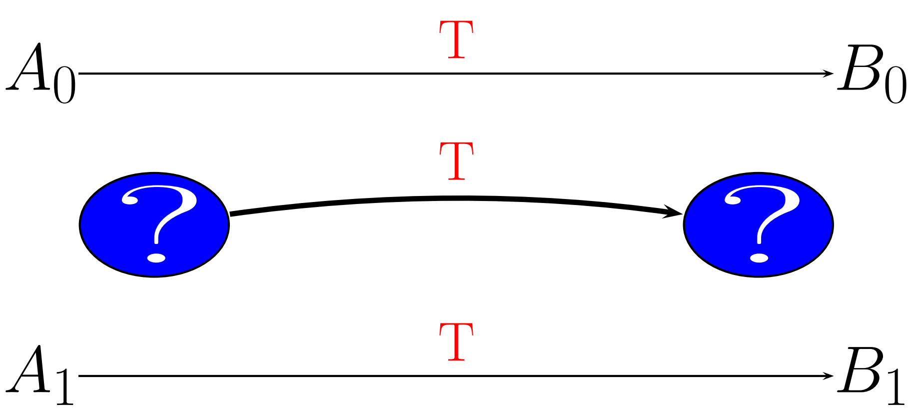


INTERPOLATION METHODS

$\overline{A} = (A_0, A_1)$ interpolation couple.

$$K(t, a; \overline{A}) = \inf_{a = a_0 + a_1} \{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\}$$

$$J(t, a; \overline{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}$$

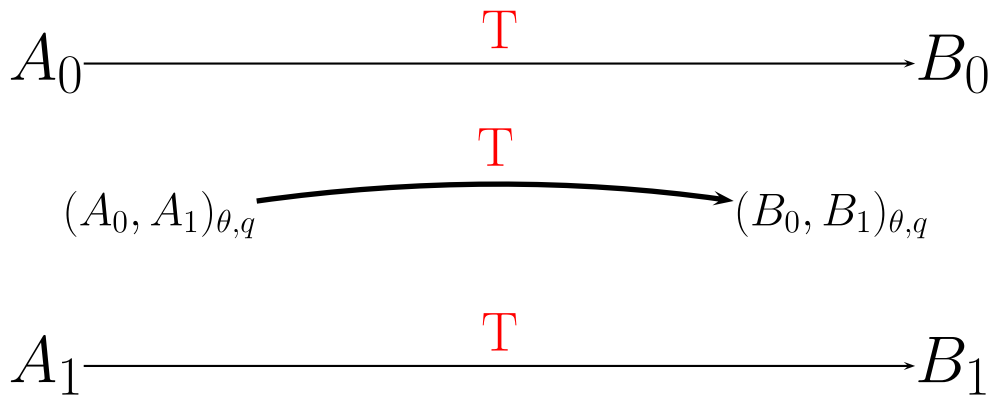
K AND J SPACES

$(A_0, A_1)_{\theta,q;J}$ consists of all $a = \sum_{-\infty}^{\infty} u_m$

$$\|a\|_{\theta,q;J} = \inf\left\{\left(\sum_{-\infty}^{\infty}\left(2^{-\theta m} J(2^m, u_m)\right)^q\right)^{\frac{1}{q}}\right\} < \infty$$

$(A_0, A_1)_{\theta,q;K}$ consists of all $a \in \sum(\overline{A})$

$$\|a\|_{\theta,q;K} = \left(\sum_{m=-\infty}^{\infty}\left(2^{-\theta m} K(2^m, a)\right)^q\right)^{\frac{1}{q}} < \infty$$



Norm Estimates

$$\boxed{\|T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}\| \leq C\|T_{A_0,B_0}\|^{1-\theta}\|T_{A_1,B_1}\|^{\theta}}$$

COMPACTNESS

Krasnoselskij Theorem, 1960.

$$\left.\begin{array}{l} L_{p_0} \xrightarrow[\text{Compact}]{\text{T}} L_{q_0} \\ \\ L_{p_1} \xrightarrow{\text{T}} L_{q_1} \end{array}\right. \Longrightarrow L_p \xrightarrow[\text{Compact}]{\text{T}} L_q$$

$1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $q_0 < \infty$.

Cobos, Kühn, Schonbek 1992 / Cwikel 1992.

$$\left.\begin{array}{l} A_0 \xrightarrow[\text{Compact}]{\text{T}} B_0 \\ \\ A_1 \xrightarrow{\text{T}} B_1 \end{array}\right. \Longrightarrow \overline{A}_{\theta,q} \xrightarrow[\text{Compact}]{\text{T}} \overline{B}_{\theta,q}$$

Interpolation Properties

of the entropy numbers

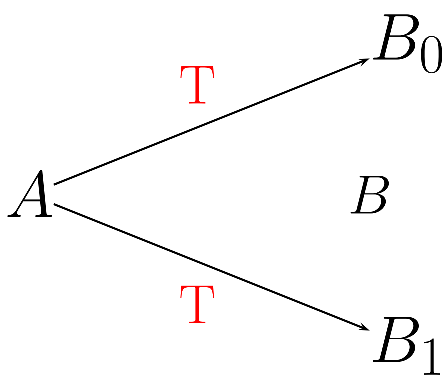
A intermediate space for $\overline{A} = (A_0, A_1)$.

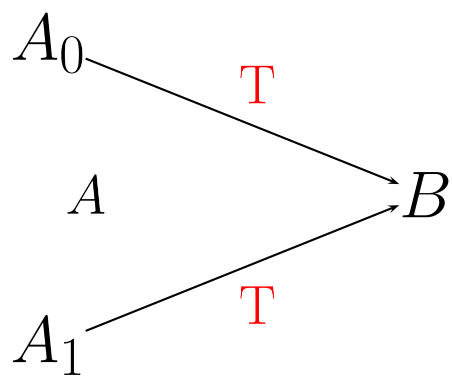
$$A \in C_K(\theta; \overline{A}) \iff \Delta(\overline{A}) \hookrightarrow A \hookrightarrow (A_0, A_1)_{\theta,\infty}$$

$$\iff \boxed{K(t,a) \leq C t^\theta \|a\|_A}$$

$$A \in C_J(\theta; \overline{A}) \iff (A_0, A_1)_{\theta,1} \hookrightarrow A \hookrightarrow \Sigma(\overline{A})$$

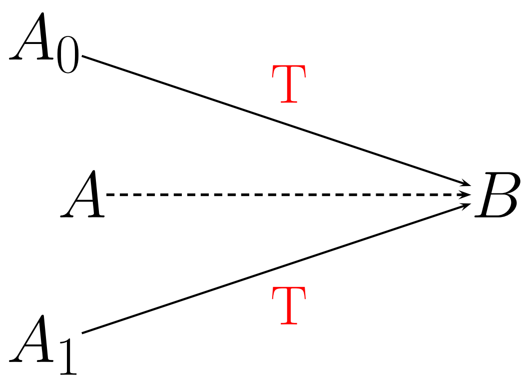
$$\boxed{\overline{A}_{\theta,q} \in C_K(\theta, q) \cap C_J(\theta, q).}$$





$B \in C_K(\theta, \overline{B})$ $\quad$ $A \in C_J(\theta, \overline{A})$

$$\boxed{e_{n+m-1}(T : A \longrightarrow B) \leq 2C e_m(T_{A_0,B_0})^{1-\theta} e_n(T_{A_0,B_0})^{\theta}}$$



$$r_0 > e_n(T : A_0 \longrightarrow B) \Longleftrightarrow T(\mathfrak{U}_{A_0}) \subset \cup_{j=1}^{2^{n-1}} B(y_j, r_0)$$

$$r_1 > e_m(T : A_1 \longrightarrow B) \Longleftrightarrow T(\mathfrak{U}_{A_1}) \subset \cup_{k=1}^{2^{m-1}} B(z_k, r_0)$$

$$(1) \quad A \in C_K(\theta, \overline{A}) \Rightarrow t^{-\theta} K(t, a) \leq C \|a\|_A$$

$$t = \tfrac{r_1}{r_0},\ a \in \mathfrak{U}_A \ \boxed{\text{then } a = a_0 + a_1}$$

$$\boxed{\|a_0\|_{A_0} + \tfrac{r_1}{r_0} \|a_1\| A_1 \leq C(1 + \varepsilon) \left(\tfrac{r_1}{r_0}\right)^{\theta}}$$

$$Ta_0 \in B(\overline{y}_j, (1+\varepsilon)Cr_0^{1-\theta}r_1^{\theta})$$

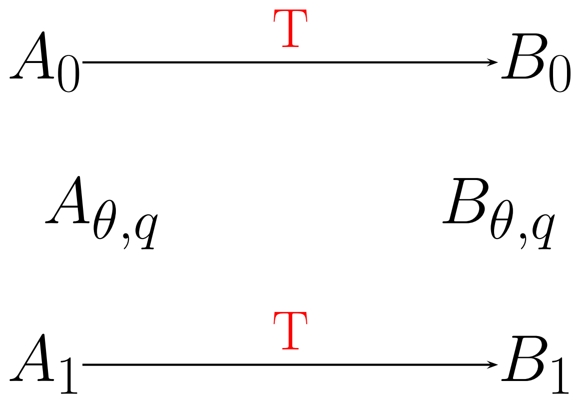
$$Ta_1 \in B(\overline{z}_k, (1+\varepsilon)Cr_0^{1-\theta}r_1^{\theta})$$

Since $Ta = Ta_0 + Ta_1$

$$Ta \in B(\overline{y}_j + \overline{z}_k, (1+\varepsilon)2Cr_0^{1-\theta}r_1^{\theta})$$

$1 \leq j \leq 2^{n-1}$ and $1 \leq k \leq 2^{m-1}$.

$$\boxed{e_{n+m-1}(T : A \longrightarrow B) \leq 2Ce_m(T_{A_0,B})^{1-\theta}e_n(T_{A_1,B})^{\theta}}$$



?

$$e_{n+m-1}(T : A_{\theta,q} \longrightarrow B_{\theta,q}) \leq 2Ce_m(T_{A_0,B_0})^{1-\theta}e_n(T_{A_1,B_1})^{\theta}$$

Measure of non-Compactness

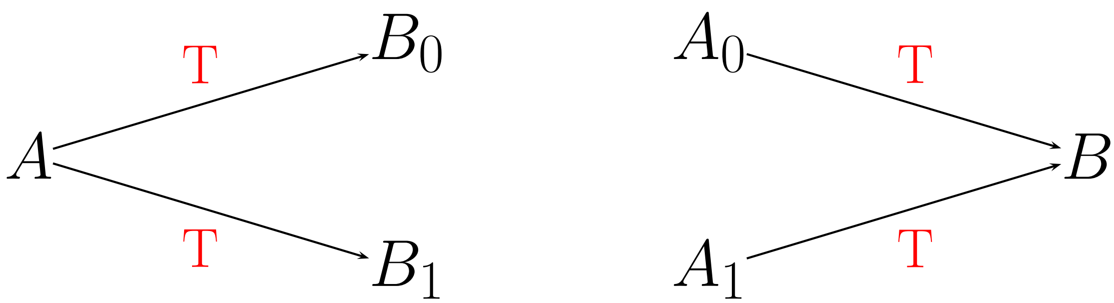
$$T : A \longrightarrow B$$

$\beta(T)$ is the infimum of all $r > 0$

$$T(\mathcal{U}_A) \subseteq \bigcup_{j=1}^{s} \{b_j + r\mathcal{U}_B\}.$$

$$\boxed{\beta(T) = \lim_n e_n(T)}$$

- $0 \leq \beta(T) \leq \|T\|$
- T is compact iff $\beta(T) = 0$



$$e_{n+m-1}(T:\overline{A}_{\theta,q}\longrightarrow\overline{B}_{\theta,q})\leq 2Ce_m(T_{A_0,B_0})^{1-\theta}e_n(T_{A_0,B_0})^{\theta}$$

$$\beta(T:\overline{A}_{\theta,q}\longrightarrow\overline{B}_{\theta,q})\leq 2C\beta(T_{A_0,B_0})^{1-\theta}\beta(T_{A_0,B_0})^{\theta}$$

$$A_0 \xrightarrow{\ \mathrm{T}\ } B_0$$

$$A_1 \xrightarrow{\ \mathrm{T}\ } B_1$$

?

$$e_{n+m-1}(T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}) \leq 2C e_m(T_{A_0,B_0})^{1-\theta} e_n(T_{A_1,B_1})^{\theta}$$

$$\beta(T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}) \leq 2C \beta(T_{A_0,B_0})^{1-\theta} \beta(T_{A_1,B_1})^{\theta}$$

Approximation Hypothesis. $\overline{B} = (B_0, B_1)$ satisfies the Approximation Hypothesis if:

Given $\varepsilon > 0$, F_0 and F_1

$$\exists P \in \mathfrak{L}(\overline{B}, \overline{B}) \text{s.t.}$$

(i) $P : B_k \longrightarrow B_k$ is compact, $k = 0$ or 1.

(ii) $P(B_k) \subset \Delta(\overline{B})$ for $k = 0, 1$.

(iii) $\|I - P\|_k \leq C_k$ and $\|x - Px\|_{B_k} < \varepsilon$

for all $x \in F_k$, $k = 0, 1$.

Example. X is a locally compact space with a positive measure μ, $1 \leq p_0, p_1 < \infty$,

$$\Big\{L_{p_0}(X, \mu), L_{p_1}(X, \mu)\Big\}$$

satisfies the Approximation Hypothesis

Persson Theorem. Let $\overline{B} = (B_0, B_1)$ with Approximation Hypothesis, then

$$\begin{array}{c} A_0 \xrightarrow[\text{Compact}]{\text{T}} B_0 \\ \\ A_1 \xrightarrow{\text{T}} B_1 \end{array} \quad \Longrightarrow \quad \overline{A}_{\theta,q} \xrightarrow[\text{Compact}]{\text{T}} \overline{B}_{\theta,q}$$

Edmunds & Teixeira.

$T : \overline{A} \longrightarrow \overline{B}$,

$\overline{B}$ with the Approximation Hypothesis, then

$$\boxed{\beta(T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}) \leq 2C\beta(T_{A_0,B_0})^{1-\theta}\beta(T_{A_1,B_1})^{\theta}}$$

Proof. Given $\varepsilon > 0 \; \exists P \in \mathfrak{L}(\overline{B}, \overline{B})$ s.t.

(i) $P : B_k \longrightarrow B_k$ is compact, $k = 0$ or 1.

(ii) $P(B_k) \subset \Delta(\overline{B})$ for $k = 0, 1$.

(iii) $\|T - PT\|_k \leq C_k \beta(T_{A_k,B_k}) + \varepsilon, \; k = 0, 1$.

$$\boxed{T = T - PT + PT}$$

$$\beta(T) \leq \beta(T - PT) + \beta(PT)$$

0

$$\beta(T - PT) \leq \|T - PT\|_{\overline{A}_{\theta,q},\overline{B}_{\theta,q}} \leq$$

$$C\|T - PT\|_{A_0,B_0}^{1-\theta} \|T - PT\|_{A_1,B_1}^{\theta} \leq$$

$$C(C_0\beta_0(T) + \varepsilon)^{1-\theta}(C_1\beta_1(T) + \varepsilon)^{\theta}.$$

$\square$

Interpolation of the measure of non-compactness by the real method

by

Fernando Cobos, Pedro Fernández-Martínez and

Antón Martínez

Theorem 1. *Let* (A_0, A_1) *and* (B_0, B_1) *be Banach couples and let* T *be a linear operator such that* $T : A_0 \longrightarrow B_0$ *and* $T : A_1 \longrightarrow B_1$ *are bounded. Then for any* $1 \leq q \leq \infty$ *and* $0 < \theta < 1$ *we have*

$$\beta_{\theta,q}(T) \leq C\beta_0(T)^{1-\theta}\beta_1(T)^{\theta}$$

where $C = 162^{\theta}(3 - 2^{\theta} - 2^{1-\theta})^{-1}$.

Studia Math. **135** (1999), 1, 25-38.

Two References

[1] F. Cobos, L.M. Fernández Cabrera, and A. Martínez. Abstract K and J spaces and measure of non-compactness. *Math. Nachr.*

[2] P. Fernández-Martínez. Interpolation of the measure of non-compactness between quasi-Banach spaces. *Rev. Mat. Complut.*, **19** (2006), 2 477-498.

$$T : \overline{A} \longrightarrow \overline{B}$$

$$\|T\|_{\mathcal{M}(\overline{A}),\mathcal{M}(\overline{B})} \leq \varphi_{\mathcal{M}}\big(\|T\|_0, \|T\|_1\big).$$

$\varphi_{\mathcal{M}}$ is:

- Homogeneous of degree 1.

- Non-decreasing in each variable.

- $\varphi_{\mathcal{M}}(t^{1-k}, t^k) \to 0$ as $t \to 0,\ k = 0, 1.$

Theorem 2. $\overline{B} = (B_0, B_1)$ *quasi-Banach couple with* A.H., *and* $\mathcal{M}$ *as above, then for any* $T : \overline{A} \longrightarrow \overline{B}$*,*

$$\beta\big(T : \mathcal{M}(\overline{A}) \longrightarrow \mathcal{M}(\overline{B})\big) \leq C\varphi_{\mathcal{M}}(\beta(T_{A_0,B_0}), \beta(T_{A_1,B_1})).$$

Parameter Function

$\rho : (0,\infty) \longrightarrow (0,\infty)$ s.t.

$$\rho(t) \text{ increases } 0 \nearrow \infty$$

$$\frac{\rho(t)}{t} \text{ decreases } \infty \searrow 0$$

$$s_\rho(\lambda) = \circ(\max\{1,\lambda\})$$

where $s_\rho(\lambda) = \sup_{u>0} \frac{\rho(\lambda u)}{\rho(u)}$.

GUSTAVSSON-PEETRE ± METHOD

$\langle \overline{A}, \rho \rangle$ collection of all $a = \sum u_m$

$$\Big\| \sum_{\mathcal{F}} \varepsilon_m 2^{jm} \frac{u_m}{\rho(2^{jm})} \Big\|_{A_j} \le C \sup_{m \in \mathbb{Z}} |\varepsilon_m|$$

hold for some constant C, any finite set $\mathcal{F}$ and $j = 0, 1$.

Example. Let $p_0 < p_1$ and $0 < \theta < 1$, then

$$\boxed{\langle L_{p_0}, L_{p_1}, \Phi \rangle = L^{\varphi}}$$

$$\Phi = x^{\theta}(\log(e + x))^{\alpha}(\log(e + \frac{1}{x}))^{\beta}$$

$$\varphi = x^{p}(\log(e + x))^{-\beta p}(\log(e + \frac{1}{x}))^{-\alpha p}$$

$1/p = (1 - \theta)/p_0 + \theta/p_1$.

Ovchinnikov Method

$$H_1(\overline{A}) = \mathrm{Corb}_{\ell_1(\frac{1}{\rho(2^n)})}\big[\ell_1, \ell_1(2^{-n})\big](\overline{A}),$$

the space of all elements $a \in \Sigma(\overline{A})$ such that

$$\sup_{\|T\|_{\overline{\ell_1},\overline{A}} \leq 1} \Big\{ \|Ta\|_{\ell_1(\frac{1}{\rho(2^n)})} \Big\} < \infty$$

where $\overline{\ell_1} = (\ell_1, \ell_1(2^{-n}))$.

$$\boxed{\beta\big(T : \mathcal{M}(\overline{A}) \longrightarrow \mathcal{M}(\overline{B})\big) \leq C\beta(T_{A_0,A_1})\, s_\rho\Big(\frac{\beta(T_{A_1,B_1})}{\beta(T_{A_0,B_0})}\Big)}$$

Generalized K and J Methods

Γ a sequence lattice with all finite sequences

$$a \in (A_0, A_1)_{\theta,q;K} \iff K(2^m, a) \in \ell_q(2^{-\theta m})$$

Assume $(\min 1, 2^m) \in \Gamma$

$$\boxed{a \in (A_0, A_1)_{\Gamma;K} \iff K(2^m, a) \in \Gamma}$$

$$a = \sum u_m \in (A_0, A_1)_{\theta,q;J} \iff J(2^m, u_m) \in \ell_q(2^{-\theta m})$$

Assume $\sup_{\|(\xi_m)\|_\Gamma \le 1} \left\{ \sum_{-\infty}^{\infty} \min 1, 2^{-m}|\xi_m| \right\}$

$$\boxed{a \in (A_0, A_1)_{\Gamma;J} \iff J(2^m, u_m) \in \Gamma}$$

$$(L_1, L_\infty)_{\theta,q} \rightsquigarrow L_{p,q}$$

$$(L_1, L_\infty)_\Gamma \rightsquigarrow L_{p,q}(logL)^\gamma$$

$$\boxed{\beta(T_{\overline{A}_{\Gamma;J},\overline{B}_{\Gamma;K}}) \leq C\beta(T_{A_0,B_0})f\Big(\frac{\beta(T_{A_1,B_1})}{\beta(T_{A_0,B_0})}\Big)}$$