About some properties of function sets, for which the condition of uniform convergence of means to expectations are not valid

In the previous works we have found necessary and sufficient conditions of uniform convergence of means to expectations for a set of limited functions $Q(x,\alpha)$ ($0 \leq Q(x,\alpha) \leq A$, x- elementary event, α - parameter):

$$P\{\sup (\alpha \in \Lambda) |E\, Q(x,\alpha) - 1/l\, \Sigma\, Q(x_i,\alpha)| > \varepsilon \} \to 0 \qquad \text{by } l \to \infty,$$

where $\mathbf{x}^l = (x_1, \ldots x_l)$ is an iid sample set of length l, for all $\varepsilon > 0$.

To formulate these conditions we proposed the following construction. Given a sample set of length l we define a subset Z of l- dimensional Euclidian space with vectors $\mathbf{z} = (z_1, \ldots z_l)$ according to the rule:

$$Z = \{ \mathbf{z} \colon \exists\, \alpha \in \Lambda\ \forall_{i=1,..,l}\, (Q(x_i,\alpha) = z_i)\}$$

Then we consider ε –extension $Y(\mathbf{x}^l, \varepsilon)$ of the set Z as unification of all open cubes, oriented along coordinates with all edges equal to ε and centres in all points of the set Z. Let $V_\varepsilon(\mathbf{x}^l)$ be the volume of this set. It is supposed that $V_\varepsilon(\mathbf{x}^l)$ is a measurable function of $\mathbf{x}^l$. Then there always exists expectation of the volume log, which we call ε–entropy of the function set $Q(x,\alpha)$ on the samples of the length l

$$H_\varepsilon = E \ln V_\varepsilon(\mathbf{x}^l),$$

and there exists asymptotic value of entropy for a symbol defined as

$$c_\varepsilon = \lim_{(l \to \infty)} H_\varepsilon / l.$$

It is always true that $\ln \varepsilon \leq \ln V_\varepsilon(\mathbf{x}^l) \leq \ln A$.

The necessary and sufficient condition of the uniform convergence is that c_ε is equal to $\ln(\varepsilon)$ for any $\varepsilon > 0$.

In the present paper we consider in more details the case when this condition is not valid, i.e

$$c_\varepsilon = \ln \varepsilon + \eta, \qquad \text{where } \eta \text{ is more than zero.}$$

The following statement is proved.

Let $c_\varepsilon = \ln \varepsilon + \eta$, ($\eta > 0$), then there exist two functions $\psi_0(x) \leq \psi_1(x)$ with the following properties:

1. $\int (\psi_1(x) - \psi_0(x))\, dP \geq \varepsilon (e^{\eta} - 1)$,

2. for any $\delta > 0$ and $l \geq 1$, almost any sequence $x_1, \ldots x_l$ and any binary sequence $\omega_1, \ldots \omega_l$ ($\omega_i = 0,1$) there exists such a value $\alpha^* \in \Lambda$, that for all i ($1 \leq i \leq l$)

$$| Q(x_i,\alpha^*) - \psi_{\omega i}(x_i)| < \delta.$$

In other words, there exists a corridor of non-zero thickness, bounded from top and bottom by the functions $\psi_1(x)$ and $\psi_0(x)$ such, that for almost any sequence $x_1,\ldots x_l$ there exists a function $Q(x,\alpha^*)$ ($\alpha^* \in \Lambda$) arbitrarily close to upper or lower bounds of the corridor in the points of sampling in arbitrary order. The thickness of the corridor does not depend on the length of the sequence.

Starting from this result it is easy to estimate the largest deviation of mean value from expectation over the class of function, and see that it does not go to zero even by unlimited growth of the sample sequence length.